FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Counting operators in Effective Field Theories

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May 16, 2023

Effective Field Theory Action

An effective approximation to a given more fundamental theory (may not be actually known) can be obtained by

(1) choosing a subset of particle fields $\{\Phi_i\}$,

(2) constructing the **most general effective action** consistent with *locality*, *Lorentz invariance* and additional *symmetries*,

$$S_{\text{eff}}[\{\Phi_i\}] = \int_{\mathcal{M}} d^{\mathsf{d}}x \left[\mathcal{L}_{\text{kin}} + \sum_j \frac{c_j}{\Lambda^{\Delta_j - \mathsf{d}}} \mathcal{O}_j\right],$$

(3) and finally determining (matching) the coefficients c_j.

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Relations Between Operators

Not all operators are independent.

Definition (Operator relations). Operators $\mathcal{O}_m$ and $\mathcal{O}_n$ are considered equivalent (denoted by $\mathcal{O}_m \sim \mathcal{O}_n$), if they satisfy:

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$$\exists\, \mathcal{O}', \Phi_j : \ \mathcal{O}_m = \mathcal{O}_n + \frac{\delta S_{\text{kin}}}{\delta \Phi_j}\mathcal{O}'$$

(b) *Integration by parts* — **IBP** $\left(\int_M \partial \cdot \mathcal{O} = \int_{\partial M} \mathcal{O} = 0\right)$

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Example (EOM and IBP).

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Definition (Operator basis). *The operator basis* $\mathcal{B}$ of the EFT is a minimal set of operators leading to all possible physical phenomena in the realm of the EFT.

In general it is hard to construct $\mathcal{B}$. An easier step is to at least count independent operators of different types.

Definition (Hilbert series). The Hilbert series is a formal series

$$H(\phi, \mathcal{D}) = \sum_{\boldsymbol{r}} \sum_{n=0}^{\infty} d_{\boldsymbol{r}n} \phi^{\boldsymbol{r}} \mathcal{D}^n,$$

where $d_{\boldsymbol{r}n} \equiv d_{r_1 \ldots r_N n} \in \mathbb{N}_0$ is the number of independent operators in the operator basis $\mathcal{B}$ of the type $\partial^n \boldsymbol{\Phi}^{\boldsymbol{r}}$.

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where $d_{\boldsymbol{r}n} \equiv d_{r_1 \ldots r_N n} \in \mathbb{N}_0$ is the number of independent operators in the operator basis $\mathcal{B}$ of the type $\partial^n \boldsymbol{\Phi}^{\boldsymbol{r}}$.

Example — N Scalar Fields $\{\Phi_i\}$ in $\mathsf{d} = 1$

(1) *No relations.* Rather trivial, because the operator basis can be easily guessed. It is freely generated by the set $\{\partial^n \Phi_i\}$ with $i = 1, \ldots, N$ and $n \in \mathbb{N}_0$. It is easy to see that every operator is obtained precisely once in the expansion of

$$\prod_{i=1}^{N} (1 + \Phi_i + \Phi_i^2 + \ldots)(1 + \partial\Phi_i + (\partial\Phi_i)^2 + \ldots) \times \\ \times (1 + \partial^2\Phi_i + (\partial^2\Phi_i)^2 + \ldots)(1 + \partial^3\Phi_i + \ldots) \cdots$$

$$= \prod_{i=1}^{N} \frac{1}{(1 - \Phi_i)(1 - \partial\Phi_i)(1 - \partial^2\Phi_i) \cdots} = \prod_{i=1}^{N} \prod_{n=0}^{\infty} \frac{1}{1 - \partial^n \Phi_i}.$$

The corresponding *free Hilbert series* is obtained by substituting (Φ_i, ∂) for their corresponding labels $(\phi_i, \mathcal{D})$,

$$H_N^{\text{free}}(\phi, \mathcal{D}) = \sum_r \sum_{n=0}^{\infty} d_{rn}^{\text{free}} \phi^r \mathcal{D}^n = \prod_{i=1}^{N} \prod_{n=0}^{\infty} \frac{1}{1 - \mathcal{D}^n \phi_i}.$$

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The kinetic Lagrangian density of N scalar fields in $\mathsf{d} = 1$ is

$$\mathcal{L}_{\text{kin}}(\{\Phi_i, \partial\Phi_i\}) \equiv \sum_{i=1}^{N} \frac{1}{2}(\partial\Phi_i)^2.$$

(2) *Only EOM relations.* Considering the kinetic Lagrangian density $\mathcal{L}_{\text{kin}}$, the (free) EOM relations are $\partial^2\Phi_i = 0$.

We even have $\partial^n\Phi_i = 0$ for $n \geq 2$, so the operator basis is now freely and finitely generated by the set $\{\Phi_i, \partial\Phi_i\}_{i=1}^{N}$, giving us the *EOM Hilbert series* of the form

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So for $\boldsymbol{r} \neq \boldsymbol{0}$ we have $d^{\text{IBP}}_{\boldsymbol{r}n} = d^{\text{free}}_{\boldsymbol{r}n} - d^{\text{free}}_{\boldsymbol{r}n-1}$ (and $d^{\text{IBP}}_{\boldsymbol{0}n} = \delta_{0n}$).

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Example — N Scalar Fields $\{\Phi_i\}$ in $\mathsf{d} = 1$

(3) *Only IBP relations.* One particular example in the case of only one field flavor would be

$$0 \sim \partial\Big((\partial^{n-1}\Phi)\Phi^k\Big) = (\partial^n\Phi)\Phi^k + k(\partial^{n-1}\Phi)(\partial\Phi)\Phi^{k-1}.$$

So for $\boldsymbol{r} \neq \mathbf{0}$ we have $d^{\text{IBP}}_{\boldsymbol{r}n} = d^{\text{free}}_{\boldsymbol{r}n} - d^{\text{free}}_{\boldsymbol{r}n-1}$ (and $d^{\text{IBP}}_{\mathbf{0}n} = \delta_{0n}$).

A simple reordering of the summation gives us

$$\sum_{\boldsymbol{r}}\sum_{n=0}^{\infty} d^{\text{IBP}}_{\boldsymbol{r}n}\boldsymbol{\phi}^{\boldsymbol{r}}\mathcal{D}^n = 1 + \sum_{\boldsymbol{r}\neq\mathbf{0}}\sum_{n=0}^{\infty}(d^{\text{free}}_{\boldsymbol{r}n} - d^{\text{free}}_{\boldsymbol{r}n-1})\boldsymbol{\phi}^{\boldsymbol{r}}\mathcal{D}^n =$$

$$1 + (1-\mathcal{D})\sum_{\boldsymbol{r}\neq\mathbf{0}}\sum_{n=0}^{\infty} d^{\text{free}}_{\boldsymbol{r}n}\boldsymbol{\phi}^{\boldsymbol{r}}\mathcal{D}^n = \mathcal{D} + (1-\mathcal{D})H^{\text{free}}_N(\mathcal{D}, \{\phi_i\}).$$

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B. Henning, X. Lu, T. Melia, and H. Murayama, Hilbert series and operator bases with derivatives in effective field theories, *Communications in Mathematical Physics*, **2015**, no. 2, DOI: 10.1007/s00220-015-2518-2. arXiv: 1507.07240 [hep-th]

Only in $\mathsf{d} = 1$ the Lorentz group is trivial and the application of the derivatives is always unambiguous.

From now on, we always assume $\mathsf{d} \geq 2$, where each derivative carries an index with non-trivial transformation properties. To construct a Lorentz invariant operator, we are forced to contract all the indices (similarly for internal symmetries).

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What about d ≥ 2?

Usually there are multiple possibilities how to contract all indices (and they rapidly grow with the number of derivatives). Together with non-trivial relations this brings *substantial complexity*.

Challenge. Try to guess the number of independent operators of the type $\partial^n \Phi^4$ for $n = 2, 4, 6, 8, 10, 12, \ldots$ in d $= 4$.

$$\partial\Phi\partial\partial\Phi\partial\partial\Phi\partial\Phi \qquad \partial\partial\Phi\partial\partial\Phi\partial\Phi\partial\Phi \qquad \partial\partial\Phi\partial\partial\Phi\partial\Phi\partial\Phi$$

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How do we obtain the Hilbert series in a more general setting, for an arbitrary dimension and field content?

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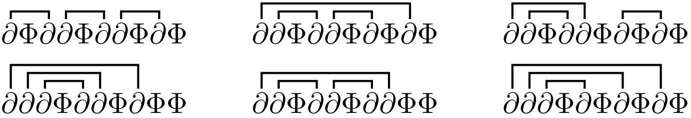

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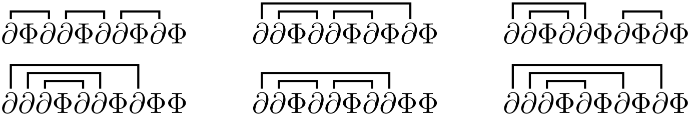

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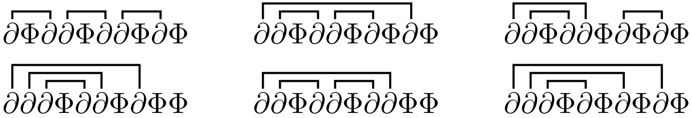

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Representations of Lie Groups

Idea: Construct the representation of all operators, and then project out only independent Lorentz invariant ones.

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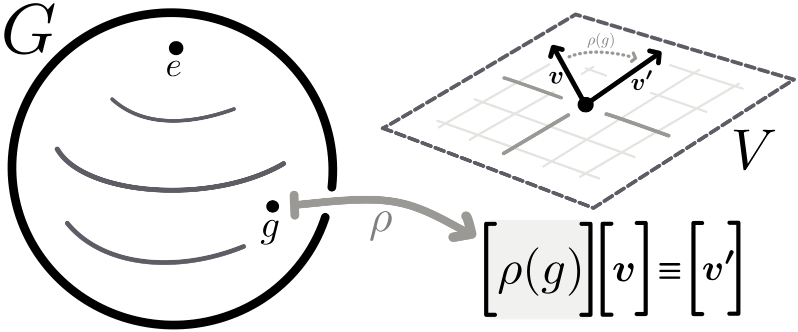

Figure: A representation ρ of a group G on a vector space V. A group element $g \in G$ is represented by a linear operator $\rho(g) \in \mathsf{GL}(V)$.

The Projection Formula

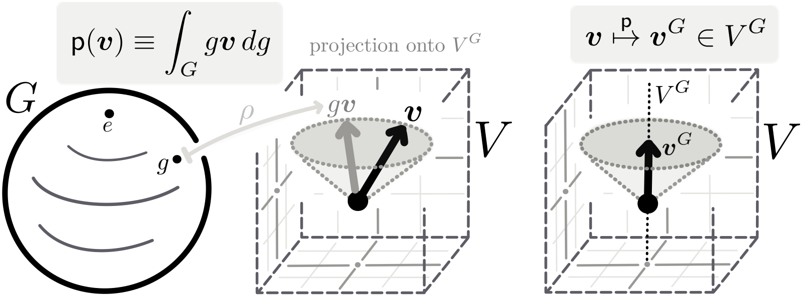

Figure: We project $\boldsymbol{v} \in V$ onto the trivial subrepresentation V^G of V. As the projection map p averages over G, the action of $g \in G$ rotates components of $\boldsymbol{v}$ in the x-y plane, leaving only $\boldsymbol{v}^G \in V^G$ pointing along the z-direction.

$$\dim V^G = \mathrm{Tr}(\mathsf{p}) = \int_G \mathrm{Tr}(g|_V)\, dg \equiv \int_G \chi_V(g)\, dg$$

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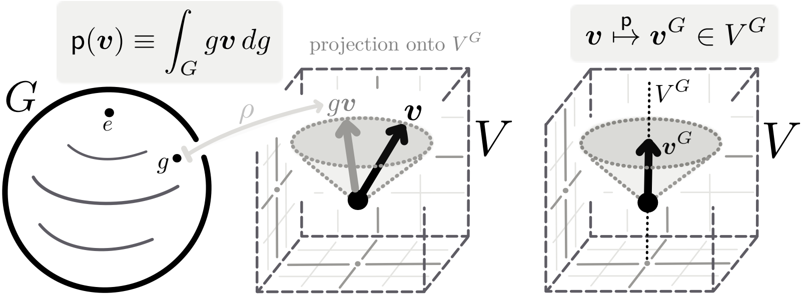

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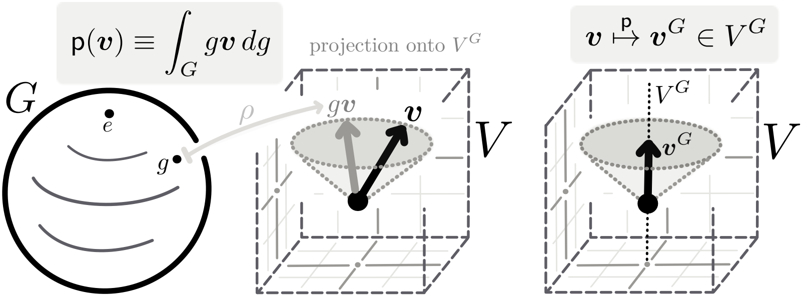

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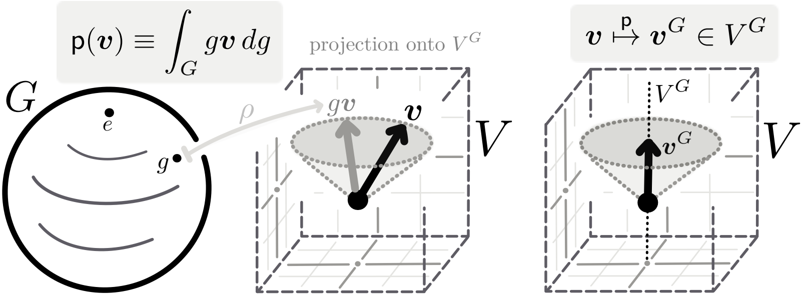

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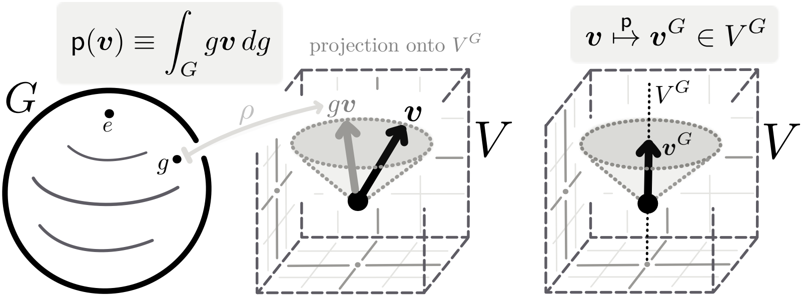

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Multiplicities and characters

More generally, we have the following formulas for multiplicities.

Theorem (Decomposition of compact Lie group representations). Let V be a representation of a compact Lie group G. Then there exists a decomposition

$$V = \bigoplus_{i=1}^{k} V_i^{\oplus a_i} \equiv V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations with multiplicities a_i given uniquely by

$$\begin{aligned} a_i &= \dim \operatorname{Hom}_G(V_i, V) \\ &= \int_G \chi_{V_i}(g^{-1})\, \chi_V(g)\, dg = \int_G \overline{\chi_{V_i}(g)}\, \chi_V(g)\, dg. \end{aligned}$$

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Strategy to compute Hilbert Series

For simplicity we will first work with a single real scalar field.

Derivation. The strategy is in the following diagram:

$$\Phi \xmapsto[\substack{\text{and EOM} \\ \partial^a\partial_a\Phi = 0}]{\bigoplus_{n=0}^{\infty} \mathcal{D}^n \partial^n \bullet} R_\Phi \simeq \begin{pmatrix} \Phi \\ \partial_a \Phi \\ \partial_{\{a_1}\partial_{a_2\}}\Phi \\ \vdots \\ \partial_{\{a_1}\cdots\partial_{a_n\}}\Phi \\ \vdots \end{pmatrix} \xmapsto[\text{since } \Phi \text{ is a boson}]{\bigoplus_{r=0}^{\infty} \phi^r S^r(\bullet)} \mathcal{J}_\Phi \mapsto \chi_{\mathcal{J}_\Phi}$$

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$$\hookrightarrow H(\phi, \mathcal{D}) = \underbrace{\int_{\mathsf{SO(d)}} \frac{1}{P(\mathcal{D}; g)} \chi_{\mathcal{J}_\Phi}(\phi, \mathcal{D}; g)\, dg}_{H_0(\phi,\mathcal{D})} + \Delta H(\phi, \mathcal{D})$$

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$$\hookrightarrow H(\phi, \mathcal{D}) = \underbrace{\int_{\mathsf{SO(d)}} \frac{1}{P(\mathcal{D}; g)} \chi_{\mathcal{J}_\Phi}(\phi, \mathcal{D}; g)\, dg}_{H_0(\phi,\mathcal{D})} + \Delta H(\phi, \mathcal{D})$$

Single Particle Graded Representation R_Φ

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(\partial_a \Phi)(\partial^a \Phi) = \frac{1}{2}(\overbracket{\partial \Phi)(\partial} \Phi) \implies \square \Phi \equiv \overbracket{\partial\partial} \Phi \sim 0$$

Derivation (Single Particle Graded Representation R_Φ)**.** We easily see that all operators composed of one Φ modulo EOM are in

$$R_\Phi = \text{Span}\begin{pmatrix} \Phi \\ \partial_a \Phi \\ \partial_{\{a_1}\partial_{a_2\}}\Phi \\ \vdots \\ \partial_{\{a_1}\cdots\partial_{a_n\}}\Phi \\ \vdots \end{pmatrix} \equiv \bigoplus_{n=0}^{\infty} \mathcal{D}^n S^{\{n\}}(\square) \equiv S^{\{\bullet\}}(\square),$$

where $_{\{\ldots\}}$ denotes the traceless symmetric part and $\square \equiv \mathbb{C}^d$ denotes the standard representation of $SO(d)$.

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Integration by parts redundancy

Operators with one free index can generate IBP relations, but only those that have nonzero divergence. Prime example which do not contribute are operators of the form

$$\partial^a \mathcal{O}_{ab} \;\text{ where }\; \mathcal{O}_{ab} \equiv \mathcal{O}_{[ab]} \quad \Longrightarrow \quad \partial^b \partial^a \mathcal{O}_{ab} = \partial^{(b}\partial^{a)}\mathcal{O}_{[ab]} = 0,$$

that is so called co-exact 1-forms. For forms we automatically have $\partial \cdot \partial \cdot \bullet = 0$, thus every co-exact form is also co-closed.

Total divergence terms are equivalent to zero by IBP relations, thus $\mathcal{K} \equiv \mathrm{Span}(\mathcal{B})$ is composed of all 0-forms (Lorentz invariants) contained in $\mathcal{J}$ modulo the co-exact ones, leading to

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Addressing IBP relations by cohomology

Derivation (Addressing IBP relations, splitting $H = H_0 + \Delta H$).

$$\begin{aligned}
\dim\mathcal{K} &= \overbrace{\dim\mathcal{J}_{[0]} - \quad \dim\mathcal{J}_{[0]\text{co-exact}}}^{\dim\mathcal{J}_{[0]\text{not co-exact}}} \\
&= \dim\mathcal{J}_{[0]} - \mathcal{D}\ \dim\mathcal{J}_{[1]\text{not co-closed}} \\
&= \dim\mathcal{J}_{[0]} - \mathcal{D}\Big(\dim\mathcal{J}_{[1]} - \dim\mathcal{J}_{[1]\text{co-closed}}\Big) \\
&= \dim\mathcal{J}_{[0]} - \mathcal{D}\Bigg(\dim\mathcal{J}_{[1]} - \underbrace{\dim\mathcal{J}_{[1]\text{co-exact}}}_{\mathcal{D}\dim\mathcal{J}_{[2]\text{not co-closed}}} - \dim\mathcal{J}_{[1]_{\substack{\text{co-closed}\\ \text{not co-exact}}}}\Bigg) \\
\text{iteratively} &\ \vdots \\
&= \underbrace{\sum_{k=0}^{\mathrm{d}} (-\mathcal{D})^k \dim\mathcal{J}_{[k]}}_{H_0} + \underbrace{\sum_{k=1}^{\mathrm{d}} (-1)^{k+1}\mathcal{D}^k \dim\mathcal{J}_{[k]_{\substack{\text{co-closed}\\ \text{not co-exact}}}}}_{\Delta H}
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Addressing IBP relations by cohomology

Derivation (Addressing IBP relations, splitting $H = H_0 + \Delta H$).

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The Master Formula

Derivation (Master Formula for H_0). Since IBP relations were addressed quite generally, we obtain *the Master Formula*

$$H_0(\phi,\mathcal{D}) = \int_{\mathsf{SO(d)}} \chi_{\bigwedge^{-}(\square)}(\mathcal{D}; g^{-1})\, \chi_{\mathcal{J}}(\phi,\mathcal{D}; g)\, dg$$
$$\equiv \int_{\mathsf{SO(d)}} \frac{1}{P(\mathcal{D}; g)}\, \chi_{\mathcal{J}}(\phi,\mathcal{D}; g)\, dg\,,$$

where

$$P(\mathcal{D}; \boldsymbol{x} \leftrightarrow g) \equiv \begin{cases} \displaystyle\prod_{i=1}^{\mathsf{r}} \frac{1}{(1-\mathcal{D}x_i)(1-\mathcal{D}/x_i)} & \text{for } \mathsf{d}=2\mathsf{r}, \\ \displaystyle\frac{1}{1-\mathcal{D}} \prod_{i=1}^{\mathsf{r}} \frac{1}{(1-\mathcal{D}x_i)(1-\mathcal{D}/x_i)} & \text{for } \mathsf{d}=2\mathsf{r}+1. \end{cases}$$

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Applications — Single Scalar Field

Bringing everything together, for a single scalar field we obtain

$$H_0(\phi,\mathcal{D}) = \int_{\mathsf{SO}(4)} \frac{1}{P(\mathcal{D};g)} \overbrace{\mathrm{PE}\Big[\phi(1-\mathcal{D}^2)P(\mathcal{D};g)\Big]}^{\chi_{\mathcal{J}_\Phi}(\phi,\mathcal{D};g)}\, dg$$

$$= \oiint_{\substack{|x_1|=1\\|x_2|=1}} (1-\mathcal{D}x_1)(1-\mathcal{D}/x_1)(1-\mathcal{D}x_2)(1-\mathcal{D}/x_2)\times$$
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With the help of `Mathematica` we obtain for $\mathsf{d} = 4$ and $\boldsymbol{r} = \mathbf{4}$:

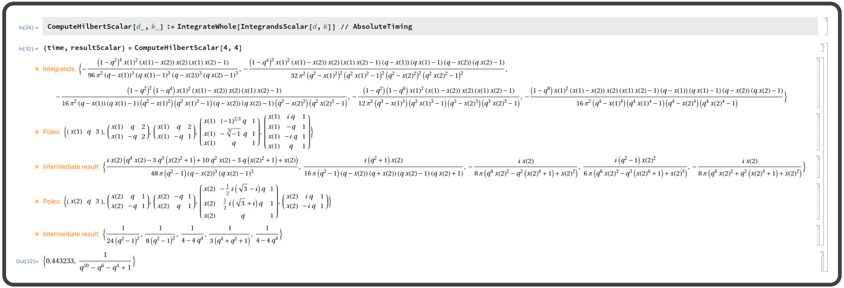

Figure: Calculation of $H_4(\mathcal{D} \leftrightarrow q)$ in the accompanying `Mathematica` notebook.

$$H_4(\mathcal{D}) \equiv H(\phi, \mathcal{D})\Big|_{\phi^4} = \frac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$$

$$= 1 + \mathcal{D}^4 + \mathcal{D}^6 + \mathcal{D}^8 + \mathcal{D}^{10} + 2\mathcal{D}^{12} + \cdots$$

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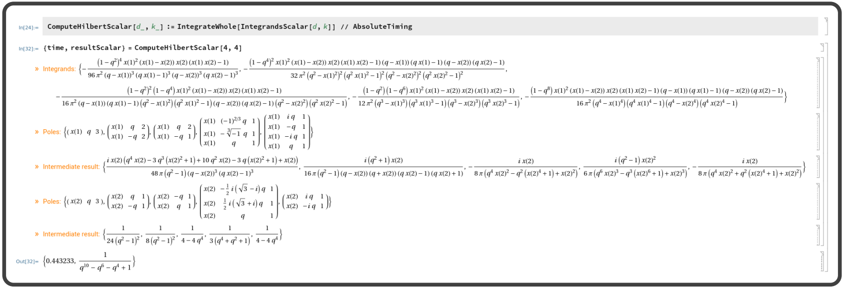

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Applications — Single Scalar Field

With the help of `Mathematica` we obtain for $\mathsf{d} = 4$ and $\boldsymbol{r} = \mathbf{4}$:

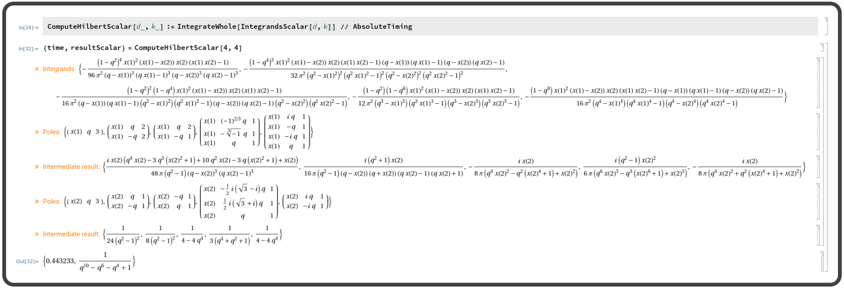

Figure: Calculation of $H_{\mathbf{4}}(\mathcal{D} \leftrightarrow q)$ in the accompanying `Mathematica` notebook.

$$H_{\mathbf{4}}(\mathcal{D}) \equiv H(\phi, \mathcal{D})\Big|_{\phi^4} = \frac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$$

$$= 1 + \mathcal{D}^4 + \mathcal{D}^6 + \mathcal{D}^8 + \mathcal{D}^{10} + 2\mathcal{D}^{12} + \cdots$$

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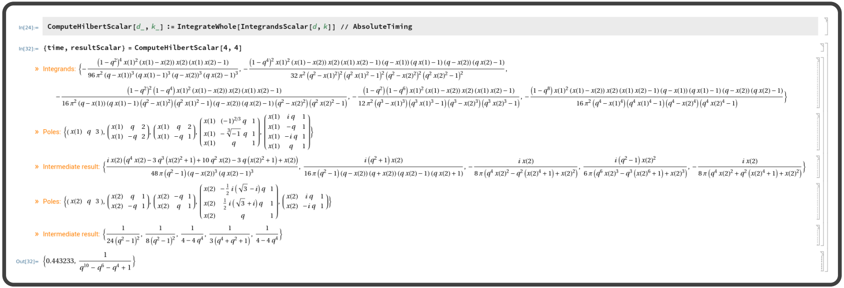

Figure: Calculation of $H_4(\mathcal{D} \leftrightarrow q)$ in the accompanying `Mathematica` notebook.

$$H_4(\mathcal{D}) \equiv H(\phi, \mathcal{D})\Big|_{\phi^4} = \frac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$$

$$= {\color{red}1} + \mathcal{D}^4 + \mathcal{D}^6 + \mathcal{D}^8 + \mathcal{D}^{10} + 2\mathcal{D}^{12} + \cdots$$

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Applications — Single Scalar Field

With the help of **Mathematica** we obtain for $\mathsf{d} = 4$ and $\boldsymbol{r} = \mathbf{4}$:

```
In[24]:= ComputeHilbertScalar[d_, k_] := IntegrateWhole[IntegrandsScalar[d, k]] // AbsoluteTiming

In[32]:= {time, resultScalar} = ComputeHilbertScalar[4, 4]
```

» Integrands: $\left\{-\frac{(1-q^2)^4 x(1)^2 (x(1)-x(2)) x(2) (x(1) x(2)-1)}{96\pi^2 (q-x(1))^3 (q x(1)-1)^3 (q-x(2))^3 (q x(2)-1)^3}, -\frac{(1-q^4)^2 x(1)^2 (x(1)-x(2)) x(2) (x(1) x(2)-1) (q-x(1)) (q x(1)-1) (q-x(2)) (q x(2)-1)}{32\pi^2 (q^2-x(1)^2)^2 (q^2 x(1)^2-1)^2 (q^2-x(2)^2)^2 (q^2 x(2)^2-1)^2}, -\frac{(1-q^2)^2 (1-q^4) x(1)^2 (x(1)-x(2)) x(2) (x(1) x(2)-1)}{16\pi^2 (q-x(1)) (q x(1)-1) (q^2-x(1)^2) (q^2 x(1)^2-1) (q-x(2)) (q x(2)-1) (q^2-x(2)^2) (q^2 x(2)^2-1)}, -\frac{(1-q^2)(1-q^6) x(1)^2 (x(1)-x(2)) x(2) (x(1) x(2)-1)}{12\pi^2 (q^3-x(1)^3)(q^3 x(1)^3-1)(q^3-x(2)^3)(q^3 x(2)^3-1)}, -\frac{(1-q^8) x(1)^2 (x(1)-x(2)) x(2) (x(1) x(2)-1) (q-x(1)) (q x(1)-1) (q-x(2)) (q x(2)-1)}{16\pi^2 (q^4-x(1)^4)(q^4 x(1)^4-1)(q^4-x(2)^4)(q^4 x(2)^4-1)}\right\}$

» Poles: $\left\{\begin{pmatrix} x(1) & q & 3 \end{pmatrix}, \begin{pmatrix} x(1) & q & 2 \\ x(1) & -q & 2 \end{pmatrix}, \begin{pmatrix} x(1) & q & 2 \\ x(1) & -q & 1 \end{pmatrix}, \begin{pmatrix} x(1) & (-1)^{2/3} q & 1 \\ x(1) & -\sqrt[3]{-1}\, q & 1 \\ x(1) & q & 1 \end{pmatrix}, \begin{pmatrix} x(1) & i q & 1 \\ x(1) & -q & 1 \\ x(1) & -i q & 1 \\ x(1) & q & 1 \end{pmatrix}\right\}$

» Intermediate result: $\left\{\frac{i\, x(2) (q^4 x(2) - 3 q^3 (x(2)^2+1) + 10 q^2 x(2) - 3 q (x(2)^2+1) + x(2))}{48\pi (q^2-1)(q-x(2))^3 (q x(2)-1)^3}, \frac{i (q^2+1) x(2)}{16\pi (q^2-1)(q-x(2))(q+x(2))(q x(2)-1)(q x(2)+1)}, -\frac{i\, x(2)}{8\pi (q^4 x(2)^2 - q^2 (x(2)^4+1) + x(2)^2)}, \frac{i (q^2-1) x(2)^2}{6\pi (q^6 x(2)^3 - q^3 (x(2)^6+1) + x(2)^3)}, -\frac{i\, x(2)}{8\pi (q^4 x(2)^2 + q^2 (x(2)^4+1) + x(2)^2)}\right\}$

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» Intermediate result: $\left\{\frac{1}{24 (q^2-1)^2}, \frac{1}{8 (q^2-1)^2}, \frac{1}{4-4q^4}, \frac{1}{3 (q^4+q^2+1)}, \frac{1}{4-4q^4}\right\}$

Out[32]= $\left\{0.443233, \frac{1}{q^{10}-q^6-q^4+1}\right\}$

Figure: Calculation of $H_4(\mathcal{D} \leftrightarrow q)$ in the accompanying **Mathematica** notebook.

$$H_4(\mathcal{D}) \equiv H(\phi, \mathcal{D})\Big|_{\phi^4} = \frac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$$

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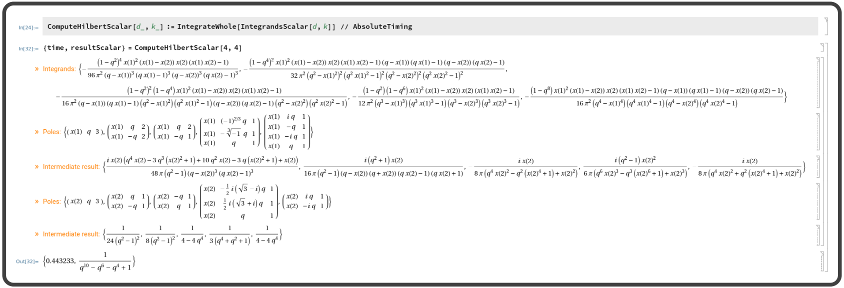

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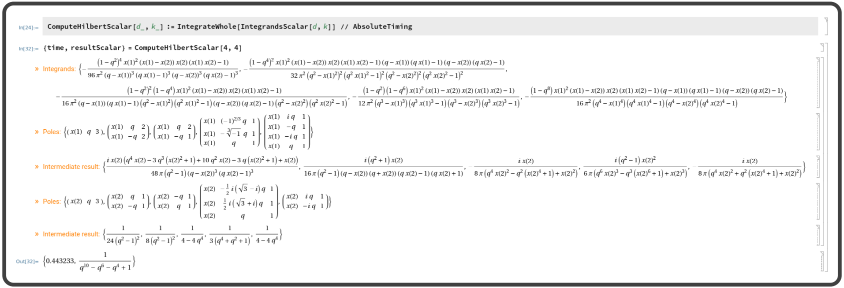

Figure: Calculation of $H_{\mathbf{4}}(\mathcal{D} \leftrightarrow q)$ in the accompanying `Mathematica` notebook.

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Applications — Single Scalar Field

d	$H_{\mathbf{4}}(\mathcal{D})$	$H_{\mathbf{5}}(\mathcal{D})$
≥ 5		$\dfrac{1+\mathcal{D}^{12}+\mathcal{D}^{14}+\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
4	$\dfrac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$\dfrac{1+\mathcal{D}^{10}+\mathcal{D}^{12}+2\mathcal{D}^{14}+2\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{22}+\mathcal{D}^{24}+\mathcal{D}^{28}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
3	$\dfrac{1+\mathcal{D}^9}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$\dfrac{1+\mathcal{D}^9+\mathcal{D}^{12}+\mathcal{D}^{13}+\mathcal{D}^{14}+2\mathcal{D}^{15}+\mathcal{D}^{16}+\mathcal{D}^{17}+\mathcal{D}^{18}+\mathcal{D}^{21}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
2	$\dfrac{1}{1-\mathcal{D}^4}$	$\dfrac{1+\mathcal{D}^{12}}{(1-\mathcal{D}^4)(1-\mathcal{D}^{12})}$

Table: The Hilbert series for a single scalar field (fixed field content $\Phi^{\mathbf{4}}$ and $\Phi^{\mathbf{5}}$).

For $\mathsf{d} = 3$ we have one additional operator for every one in $\mathsf{d} = 4$, but with 9 more derivatives. This corresponds to the operator

$$\varepsilon^{abc}(\partial\partial\partial\partial_a\Phi)(\partial\partial\partial_b\Phi)(\partial\partial_c\Phi)(\Phi).$$

Applications — Single Scalar Field

d	$H_{\mathbf{4}}(\mathcal{D})$	$H_{\mathbf{5}}(\mathcal{D})$
≥ 5		$\dfrac{1+\mathcal{D}^{12}+\mathcal{D}^{14}+\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
4	$\dfrac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$\dfrac{1+\mathcal{D}^{10}+\mathcal{D}^{12}+2\mathcal{D}^{14}+2\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{22}+\mathcal{D}^{24}+\mathcal{D}^{28}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
3	$\dfrac{1+\mathcal{D}^9}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$\dfrac{1+\mathcal{D}^9+\mathcal{D}^{12}+\mathcal{D}^{13}+\mathcal{D}^{14}+2\mathcal{D}^{15}+\mathcal{D}^{16}+\mathcal{D}^{17}+\mathcal{D}^{18}+\mathcal{D}^{21}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
2	$\dfrac{1}{1-\mathcal{D}^4}$	$\dfrac{1+\mathcal{D}^{12}}{(1-\mathcal{D}^4)(1-\mathcal{D}^{12})}$

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Applications — Single Scalar Field

d	$H_{\mathbf{6}}(\mathcal{D})$
≥ 6	$\frac{1+2\mathcal{D}^{10}+5\mathcal{D}^{12}+7\mathcal{D}^{14}+9\mathcal{D}^{16}+11\mathcal{D}^{18}+13\mathcal{D}^{20}+14\mathcal{D}^{22}+21\mathcal{D}^{24}+24\mathcal{D}^{26}+28\mathcal{D}^{28}+32\mathcal{D}^{30}+26\mathcal{D}^{32}+22\mathcal{D}^{34}+13\mathcal{D}^{36}+7\mathcal{D}^{38}+3\mathcal{D}^{40}+\mathcal{D}^{42}+\mathcal{D}^{44}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)^3(1-\mathcal{D}^{10})^2(1-\mathcal{D}^{12})}$
5	$\frac{1+2\mathcal{D}^{10}+5\mathcal{D}^{12}+7\mathcal{D}^{14}+\mathcal{D}^{15}+9\mathcal{D}^{16}+\mathcal{D}^{17}+11\mathcal{D}^{18}+3\mathcal{D}^{19}+13\mathcal{D}^{20}+7\mathcal{D}^{21}+14\mathcal{D}^{22}+13\mathcal{D}^{23}+21\mathcal{D}^{24}+22\mathcal{D}^{25}+24\mathcal{D}^{26}+26\mathcal{D}^{27}+28\mathcal{D}^{28}+32\mathcal{D}^{29}+32\mathcal{D}^{30}+28\mathcal{D}^{31}+26\mathcal{D}^{32}+24\mathcal{D}^{33}+22\mathcal{D}^{34}+21\mathcal{D}^{35}+13\mathcal{D}^{36}+14\mathcal{D}^{37}+7\mathcal{D}^{38}+13\mathcal{D}^{39}+3\mathcal{D}^{40}+11\mathcal{D}^{41}+\mathcal{D}^{42}+9\mathcal{D}^{43}+\mathcal{D}^{44}+7\mathcal{D}^{45}+5\mathcal{D}^{47}+2\mathcal{D}^{49}+\mathcal{D}^{59}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)^3(1-\mathcal{D}^{10})^2(1-\mathcal{D}^{12})}$
4	$\frac{1+3\mathcal{D}^{10}+6\mathcal{D}^{12}+11\mathcal{D}^{14}+17\mathcal{D}^{16}+22\mathcal{D}^{18}+31\mathcal{D}^{20}+36\mathcal{D}^{22}+48\mathcal{D}^{24}+53\mathcal{D}^{26}+58\mathcal{D}^{28}+58\mathcal{D}^{30}+48\mathcal{D}^{32}+38\mathcal{D}^{34}+23\mathcal{D}^{36}+14\mathcal{D}^{38}+6\mathcal{D}^{40}+4\mathcal{D}^{42}+2\mathcal{D}^{44}+\mathcal{D}^{46}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)^3(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
3	$\frac{1+\mathcal{D}^{8}+2\mathcal{D}^{9}+2\mathcal{D}^{10}+2\mathcal{D}^{11}+3\mathcal{D}^{12}+5\mathcal{D}^{13}+4\mathcal{D}^{14}+6\mathcal{D}^{15}+5\mathcal{D}^{16}+6\mathcal{D}^{17}+6\mathcal{D}^{18}+6\mathcal{D}^{19}+5\mathcal{D}^{20}+6\mathcal{D}^{21}+6\mathcal{D}^{22}+5\mathcal{D}^{23}+6\mathcal{D}^{24}+6\mathcal{D}^{25}+6\mathcal{D}^{26}+5\mathcal{D}^{27}+6\mathcal{D}^{28}+4\mathcal{D}^{29}+5\mathcal{D}^{30}+3\mathcal{D}^{31}+2\mathcal{D}^{32}+2\mathcal{D}^{33}+2\mathcal{D}^{34}+\mathcal{D}^{35}+\mathcal{D}^{43}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
2	$\frac{1+\mathcal{D}^4+\mathcal{D}^6+2\mathcal{D}^8+\mathcal{D}^{10}+3\mathcal{D}^{12}+3\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{22}}{(1-\mathcal{D}^8)(1-\mathcal{D}^{12})^2}$

Table: The Hilbert series for a single scalar field, with a fixed field content $\Phi^{\mathbf{6}}$.

Applications — Single Scalar Field

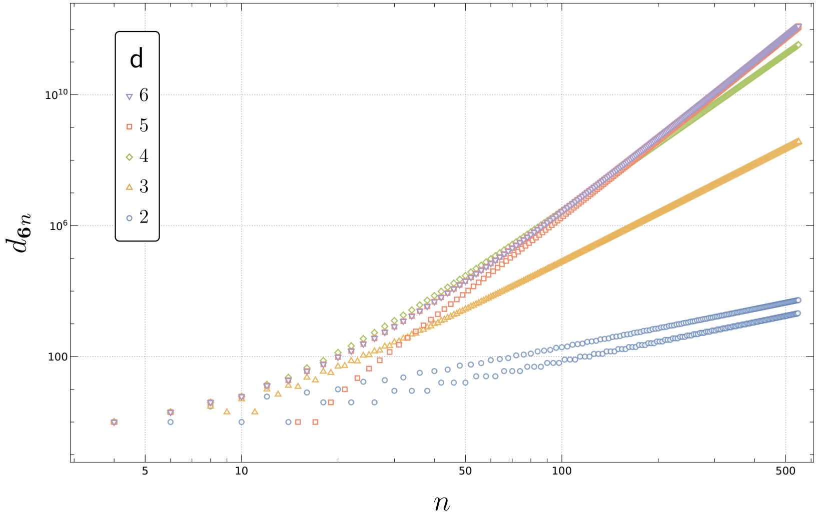

Figure: Log-log plot of the coefficients of $H_{\mathbf{6}}(\mathcal{D})$ in $\mathsf{d} = 2, \ldots, 6$.

Summary and generalization

Our starting data for any EFT are:

- Particle fields $\{\Phi_i\}$ together with their representations under the Lorentz group $\mathsf{SO(d)}$ and the internal group G.
- EOM generated from the kinetic Lagrangian density $\mathcal{L}_{\text{kin}}$.
- Possibly some other constraints.

The Hilbert series is then calculated as:

$$\{\Phi_i\} \xmapsto[\text{and EOM}]{\bigoplus_{n=0}^{\infty} \mathcal{D}^n \partial^n \bullet} \left\{ \underbrace{R_{\mathsf{SO(d)},\Phi_i} \otimes R_{G,\Phi_i}}_{R_{\Phi_i}} \right\} \xmapsto[\Phi_i \text{ is a } \substack{\text{boson} \\ \text{fermion}}]{\bigotimes_i \left(\bigoplus_{r=0}^{\infty} \phi_i^r \, \substack{S^r(\bullet_i) \\ \bigwedge^r(\bullet_i)} \right)} \mathcal{J} \mapsto \chi_{\mathcal{J}}$$

the Projection Formula + Weyl integration formula

$$\hookrightarrow H(\phi, \mathcal{D}) = \underbrace{\int_{\mathsf{SO(d)} \times G} \frac{1}{P(\mathcal{D}; g)} \chi_{\mathcal{J}}(\phi, \mathcal{D}; g) \, dg}_{H_0(\phi, \mathcal{D})} + \Delta H(\phi, \mathcal{D})$$

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the Projection Formula + Weyl integration formula

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Summary and generalization

Our starting data for any EFT are:

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It is useful to decompose representations we obtain to smaller pieces, some of which will be zero by usage of the relations.

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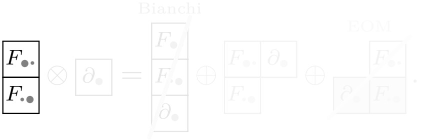

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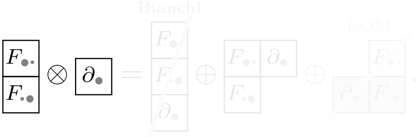

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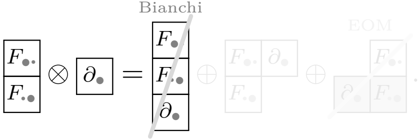

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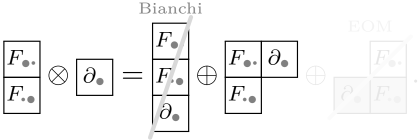

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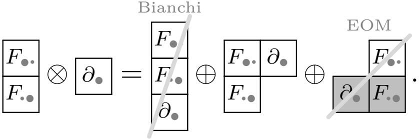

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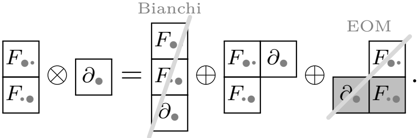

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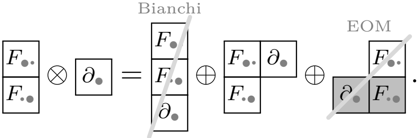

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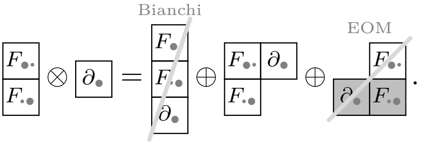

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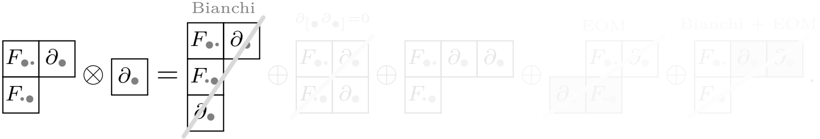

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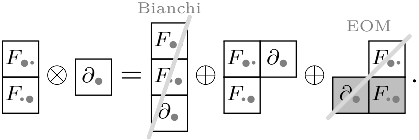

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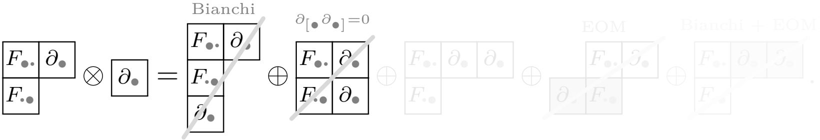

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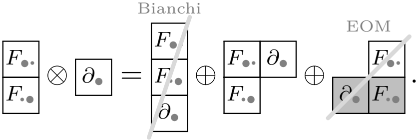

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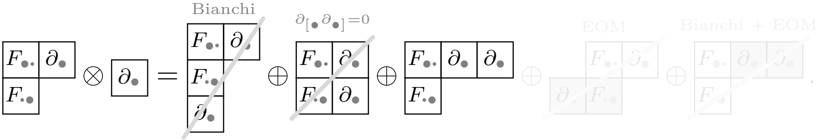

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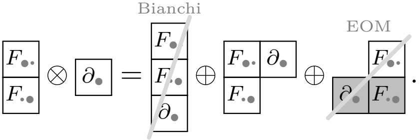

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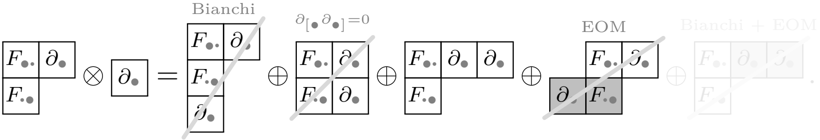

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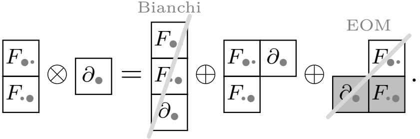

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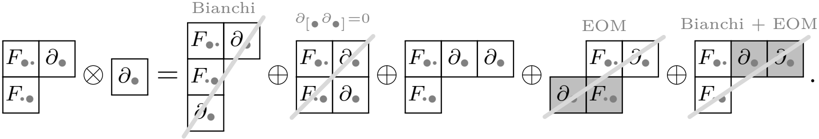

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Calculation (Single Particle Graded Representation R_F for the electromagnetic field)**.** We obtain R_F of the form

$$R_F = \mathrm{Span}\begin{pmatrix} F_{[ab]} \\ \partial_{\{a_1}F_{[a\}b]} \\ \partial_{\{a_1}\partial_{a_2}F_{[a\}b]} \\ \vdots \\ \partial_{\{a_1}\cdots\partial_{a_n}F_{[a\}b]} \\ \vdots \end{pmatrix} \equiv \mathcal{D}\,\yng(1,1) \oplus \mathcal{D}^2\,\yng(2,1) \oplus \mathcal{D}^3\,\yng(3,1) \oplus \cdots$$

$$\cdots \oplus \mathcal{D}^n\,\overbrace{\yng(1)\yng(1)\cdots\yng(1)}^{n}{}_{\yng(1)} \oplus \cdots$$

with the graded character ($F_{ab} = \mathrm{d}_a A_b$ contains one derivative)

$$\chi_{R_F}(\mathcal{D};\boldsymbol{x}) \equiv \mathcal{D}\chi_{\yng(1,1)}(\boldsymbol{x}) + \mathcal{D}^2\chi_{\yng(2,1)}(\boldsymbol{x}) + \mathcal{D}^3\chi_{\yng(3,1)}(\boldsymbol{x}) + \cdots$$

$$= \frac{\Big((\mathcal{D}-\mathcal{D}^3)\chi_{\square}(\boldsymbol{x}) - (1-\mathcal{D}^4)\Big)P(\mathcal{D};\boldsymbol{x}) + 1}{\mathcal{D}}.$$

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d	$\frac{1}{\mathcal{D}^4} H_{F^4}(\mathcal{D})$	miscount	$\frac{1}{\mathcal{D}^5} H_{0,F^5}(\mathcal{D})$
≥ 10	$\frac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\cdots$
9	$\frac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$-\mathcal{D}$	$\cdots$
8	$\frac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$+1$	$\cdots$
7	$\frac{\frac{1}{\mathcal{D}}+2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$-\frac{1}{\mathcal{D}}$	$\frac{\mathcal{D}^3+4\mathcal{D}^4+4\mathcal{D}^5+16\mathcal{D}^6+10\mathcal{D}^7+39\mathcal{D}^8+17\mathcal{D}^9+69\mathcal{D}^{10}+28\mathcal{D}^{11}+99\mathcal{D}^{12}+36\mathcal{D}^{13}+125\mathcal{D}^{14}+41\mathcal{D}^{15}+135\mathcal{D}^{16}+43\mathcal{D}^{17}+126\mathcal{D}^{18}+38\mathcal{D}^{19}+105\mathcal{D}^{20}+28\mathcal{D}^{21}+73\mathcal{D}^{22}+19\mathcal{D}^{23}+41\mathcal{D}^{24}+10\mathcal{D}^{25}+19\mathcal{D}^{26}+2\mathcal{D}^{27}+5\mathcal{D}^{28}-2\mathcal{D}^{30}-\mathcal{D}^{31}-2\mathcal{D}^{32}-\mathcal{D}^{33}+\mathcal{D}^{37}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
6	$\frac{2+3\mathcal{D}^2+2\mathcal{D}^4+\mathcal{D}^8}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\frac{1+4\mathcal{D}^2+13\mathcal{D}^4+34\mathcal{D}^6+73\mathcal{D}^8+121\mathcal{D}^{10}+168\mathcal{D}^{12}+210\mathcal{D}^{14}+226\mathcal{D}^{16}+213\mathcal{D}^{18}+182\mathcal{D}^{20}+131\mathcal{D}^{22}+79\mathcal{D}^{24}+42\mathcal{D}^{26}+16\mathcal{D}^{28}+\mathcal{D}^{30}-\mathcal{D}^{32}-\mathcal{D}^{36}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
5	$\frac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\frac{\mathcal{D}+5\mathcal{D}^3+4\mathcal{D}^4+16\mathcal{D}^5+16\mathcal{D}^6+30\mathcal{D}^7+36\mathcal{D}^8+51\mathcal{D}^9+63\mathcal{D}^{10}+73\mathcal{D}^{11}+89\mathcal{D}^{12}+92\mathcal{D}^{13}+110\mathcal{D}^{14}+103\mathcal{D}^{15}+117\mathcal{D}^{16}+103\mathcal{D}^{17}+108\mathcal{D}^{18}+91\mathcal{D}^{19}+88\mathcal{D}^{20}+71\mathcal{D}^{21}+59\mathcal{D}^{22}+49\mathcal{D}^{23}+32\mathcal{D}^{24}+27\mathcal{D}^{25}+13\mathcal{D}^{26}+12\mathcal{D}^{27}+2\mathcal{D}^{28}+3\mathcal{D}^{29}-3\mathcal{D}^{30}-2\mathcal{D}^{32}-\mathcal{D}^{33}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
4	$\frac{3+5\mathcal{D}^2+\mathcal{D}^4-2\mathcal{D}^6}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\frac{2\left(2\mathcal{D}^4+7\mathcal{D}^6+17\mathcal{D}^8+28\mathcal{D}^{10}+35\mathcal{D}^{12}+42\mathcal{D}^{14}+39\mathcal{D}^{16}+28\mathcal{D}^{18}+18\mathcal{D}^{20}+4\mathcal{D}^{22}-7\mathcal{D}^{24}-8\mathcal{D}^{26}-7\mathcal{D}^{28}-7\mathcal{D}^{30}-\mathcal{D}^{32}+2\mathcal{D}^{34}\right)}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
3	$\frac{1+\mathcal{D}^2+\mathcal{D}^5-\mathcal{D}^6}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\frac{\mathcal{D}^5+2\mathcal{D}^7+\mathcal{D}^8+\mathcal{D}^9+2\mathcal{D}^{10}+\mathcal{D}^{11}+\mathcal{D}^{12}+\mathcal{D}^{13}+\mathcal{D}^{14}+2\mathcal{D}^{16}-\mathcal{D}^{17}+\mathcal{D}^{25}-\mathcal{D}^{26}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$

Table: The Hilbert series for the electromagnetic field (with F^4 and F^5).

Summary and generalization

Our starting data for any EFT are:

- Particle fields $\{\Phi_i\}$ together with their representations under the Lorentz group $\mathsf{SO(d)}$ and the internal group G.
- EOM generated from the kinetic Lagrangian density $\mathcal{L}_{\text{kin}}$.
- Possibly some other constraints.

The Hilbert series is then calculated as:

$$\{\Phi_i\} \xmapsto[\text{and EOM}]{\bigoplus_{n=0}^{\infty} \mathcal{D}^n \partial^n \bullet} \Big\{ \underbrace{R_{\mathsf{SO(d)},\Phi_i} \otimes R_{G,\Phi_i}}_{R_{\Phi_i}} \Big\} \xmapsto[\Phi_i \text{ is a } \substack{\text{boson} \\ \text{fermion}}]{\bigotimes_i \left(\bigoplus_{r=0}^{\infty} \phi_i^r \, \substack{S^r(\bullet_i) \\ \bigwedge^r(\bullet_i)} \right)} \mathcal{J} \longmapsto \chi_{\mathcal{J}}$$

$$\xrightarrow{\text{the Projection Formula} + \text{Weyl integration formula}}$$

$$H(\phi, \mathcal{D}) = \underbrace{\int_{\mathsf{SO(d)} \times G} \frac{1}{P(\mathcal{D}; g)} \chi_{\mathcal{J}}(\phi, \mathcal{D}; g)\, dg}_{H_0(\phi, \mathcal{D})} + \Delta H(\phi, \mathcal{D})$$

Graded representations

It is hopeless to work with one operator at a time, not only efficiency-wise, but also due to IBP relations between them.

Definition (Graded representation). Representation V of G is called a *graded representation* if it has the form of

$$V \equiv \bigoplus_{n=0}^{\infty} t^n V_n,$$

where V_n is a finite-dimensional representation $\forall n \in \mathbb{N}_0$.

Example (Tensor, symmetric, and exterior graded representations). Let V be a representation of G. We define the *tensor*, *symmetric*, and *exterior graded representations* of V, respectively, as

$$T(V) \equiv \bigoplus_{n=0}^{\infty} t^n V^{\otimes n}, \quad S(V) \equiv \bigoplus_{n=0}^{\infty} t^n S^n(V), \quad \bigwedge(V) \equiv \bigoplus_{n=0}^{\dim V} t^n \bigwedge^n(V).$$

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Remark ($H(\phi, \mathcal{D})$ as a graded dimension of $\mathcal{K} = \operatorname{Span} \mathcal{B}$)**.** The space $\operatorname{Span} \mathcal{B}$ can be understood as a graded representation of the Lorentz (and possibly gauge) group. Every graded piece is composed of trivial representations, since all operators in $\mathcal{B}$ must be invariant. If we choose appropriate grading, we recognize that

$$\dim_{(\phi,\mathcal{D})} \mathcal{K} = H(\phi, \mathcal{D}).$$

The coefficients of $H(\phi, \mathcal{D})$ are thus denoted as d_{rn}, because they are the dimensions of the corresponding graded pieces.

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Graded characters

Proposition (Selected graded characters). The following graded characters of are given by:

$$\chi_{T(V)}(t;g) = \sum_{n=0}^{\infty} t^n \chi_{V^{\otimes n}}(g) = \frac{1}{1 - t\,\mathrm{Tr}(g|_V)} \equiv \frac{1}{1 - t\chi_V(g)}$$

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Remark. Graded characters are reminiscent of *partition functions* found in statistical mechanics (the grand-canonical partition functions for Bose-Einstein/Fermi-Dirac ideal quantum gases)

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Remark (Plethystic Exponential). For a function $\alpha(t_1, \ldots, t_k)$ with $\alpha(0, \ldots, 0) = 0$ we have the (fermionic) Plethystic Exponential

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For two functions α, β (either one can be bosonic or fermionic) the Plethystic Exponential satisfies the *sum-to-product* property

$$\mathrm{PE}_f[\alpha + \beta] = \mathrm{PE}_f[\alpha]\, \mathrm{PE}_f[\beta].$$

One simple example is $\alpha(t, q) \equiv t$ and $\beta(t, q) \equiv q$, for which it generates all antisymmetric combinations of the variables, that is

$$\mathrm{PE}_f[t + q] = \begin{cases} \frac{1}{(1-t)(1-q)} = 1 + t + q + t^2 + tq + q^2 + \cdots, \\ (1+t)(1+q) = 1 + t + q + tq. \end{cases}$$

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For two functions α, β (either one can be bosonic or fermionic) the Plethystic Exponential satisfies the *sum-to-product* property

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One simple example is $\alpha(t, q) \equiv t$ and $\beta(t, q) \equiv q$, for which it generates all antisymmetric combinations of the variables, that is

$$\mathrm{PE}_f[t + q] = \begin{cases} \frac{1}{(1-t)(1-q)} = 1 + t + q + t^2 + tq + q^2 + \cdots, \\ (1+t)(1+q) = 1 + t + q + tq. \end{cases}$$

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Weyl integration formula

Theorem (Weyl integration formula)**.** Let f be a class function on a connected compact Lie group G of rank r with a maximal torus T parametrized by $\boldsymbol{x} \equiv (x_1, \ldots, x_{\mathsf{r}})$. Then we have

$$\int_G f(g)\, dg = \oint\!\!\!\oint_{|x_i|=1} f(\boldsymbol{x}) \underbrace{\left[\prod_{\boldsymbol{\alpha} \in \mathrm{R}_+(G)} (1 - \boldsymbol{x}^{\boldsymbol{\alpha}}) \right]}_{\mathfrak{D}_G^+(\boldsymbol{x})} \left[\prod_i \frac{dx_i}{2\pi i x_i} \right],$$

where $\mathrm{R}_+(G)$ is the set of so-called *positive roots*.

Specifically, for $G = \mathsf{SO}(\mathsf{d})$ we have explicit forms

$$\mathfrak{D}^+_{\mathsf{SO}(\mathsf{d})}(\boldsymbol{x}) = \begin{cases} \displaystyle\prod_{1 \le i < j \le \mathsf{r}} (1 - x_i x_j)(1 - x_i/x_j) & \text{for } \mathsf{d} = 2\mathsf{r}, \\ \displaystyle\prod_{i=1}^{\mathsf{r}} (1 - x_i) \prod_{1 \le i < j \le \mathsf{r}} (1 - x_i x_j)(1 - x_i/x_j) & \text{for } \mathsf{d} = 2\mathsf{r} + 1. \end{cases}$$

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Maximal torus of $\mathsf{SO(d)}$

The maximal torus of $\mathsf{SO}(2\mathsf{r}+1)$ is

$$T_{\mathsf{SO(d)}} = \underbrace{\mathsf{SO}(2) \times \cdots \times \mathsf{SO}(2)}_{\mathsf{r}} \cong (S^1)^{\mathsf{r}}.$$

In the standard representation $\square \equiv \mathbb{C}^{\mathsf{d}} \equiv \mathbb{C}^{2\mathsf{r}+1}$ we have

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Alternatively, we can parametrize by r complex variables on the unit circle, namely by $\boldsymbol{x} \equiv (x_1, \ldots, x_{\mathsf{r}}) \equiv (e^{i\theta_1}, \ldots, e^{i\theta_{\mathsf{r}}})$.

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Characters of SO(d)

Since characters are *class functions*, and every element can be conjugated to the maximal torus, to evaluate $\chi(g)$ it is enough to specify it for any corresponding torus element $\boldsymbol{x} \leftrightarrow g$.

Using the parametrization introduced previously we obtain

$$\chi_{\square}(\boldsymbol{x}) = \begin{cases} 2\sum_{i=1}^{\mathsf{r}} \cos(\theta_i) = \sum_{i=1}^{\mathsf{r}} \left(x_i + \frac{1}{x_i} \right) & \text{for } \mathsf{d} = 2\mathsf{r}, \\ 1 + 2\sum_{i=1}^{\mathsf{r}} \cos(\theta_i) = 1 + \sum_{i=1}^{\mathsf{r}} \left(x_i + \frac{1}{x_i} \right) & \text{for } \mathsf{d} = 2\mathsf{r}+1. \end{cases}$$

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Projection factor

Derivation (Projection factor $1/P(\mathcal{D};g)$ addressing IBP relations).
We can obtain a nice alternative expression as

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\begin{aligned}
H_0(\phi,\mathcal{D}) &\equiv \sum_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \dim_{(\phi,\mathcal{D})} \mathcal{J}_{[k]} \\
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&= \dim_{(\phi,\mathcal{D})} \mathrm{Hom}_{\mathsf{SO(d)}}\left(\bigoplus_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \bigwedge^k(\square), \mathcal{J}\right) \\
&\equiv \dim_{(\phi,\mathcal{D})} \mathrm{Hom}_{\mathsf{SO(d)}}\left(\bigwedge^-(\square), \mathcal{J}\right),
\end{aligned}
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where $\bigwedge^-(\square)$ is the exterior graded representation of $\square$, but with alternating signs in the grading. Its graded character is

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&\equiv \sum_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \dim_{(\phi,\mathcal{D})} \mathrm{Hom}_{\mathsf{SO(d)}}\Big(\bigwedge^k(\Box), \mathcal{J}\Big) \\
&= \dim_{(\phi,\mathcal{D})} \mathrm{Hom}_{\mathsf{SO(d)}}\Big(\bigoplus_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \bigwedge^k(\Box), \mathcal{J}\Big) \\
&\equiv \dim_{(\phi,\mathcal{D})} \mathrm{Hom}_{\mathsf{SO(d)}}\Big(\bigwedge^-(\Box), \mathcal{J}\Big),
\end{aligned}
$$

where $\bigwedge^-(\Box)$ is the exterior graded representation of $\Box$, but with alternating signs in the grading. Its graded character is

$$
\chi_{\bigwedge^-(\Box)}(\mathcal{D}; g) \equiv \det_{\Box}(1 - \mathcal{D}g) = \frac{1}{\chi_{S(\Box)}(\mathcal{D}; g)} \equiv \frac{1}{P(\mathcal{D}; g)}.
$$