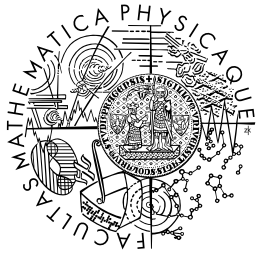

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Counting operators in Effective Field Theories

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J.D.

Title: Counting operators in Effective Field Theories

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Abstract: When applying physical calculations to the real world, it is always necessary to make some approximations. In the context of field theories, in particular the quantum field theories used in particle physics, a systematic treatment of this problem is provided by the framework of Effective Field Theories. Working in a given energy range, it is enough to consider only a couple of particles, while the existence of the rest only contributes some corrections. This is reflected in the structure of Effective Field Theory Lagrangians given as perturbation series, which are in general composed of all possible operators consistent with locality, unitarity, and symmetry assumptions. Obviously, it is advantageous to work with a minimal set of operators covering all physical phenomena, however the construction of the operator basis is very hard in general. Simpler step is to at least count all independent operators with a given number of derivatives and particle fields. To this end, the generating function (termed the Hilbert series) is introduced, with coefficients precisely being the number of independent operators of a given type. One can consider only scalar operators, that is Lorentz and gauge invariant operators, moreover consistent with other additional symmetries. To count independent operators it is furthermore necessary to handle various relations between them, induced namely by equations of motion, integration by parts, and finite-dimensionality conditions. It turns out that by virtue of understanding the transformation properties of operators, group theory together with the representation theory provide useful tools to calculate the Hilbert series. The main idea is to construct all possible operators along with corresponding group characters (which characterize their transformation properties), and project out only independent scalar operators using the character orthogonality. Big part of this thesis is focused on developing the important theory of Lie group representations, which is later necessary to derive the formula for the Hilbert series. Finally, the group theoretical formalism is applied to count operators for a single scalar field, and also a more complex case of the electromagnetic field.

Keywords: Classical field theory, Effective field theory, Group theory, Lie groups, Representation theory, Schur character orthogonality relations, Hilbert series

Contents

Introduction

Every physical calculation applied to the real world is only an approximation. This is mainly consequence of two factors — we can not measure and prepare initial states with infinite precision, and we do not posses the *exact* theory. Even if we did, it is almost impossible to calculate anything without making some mathematical shortcuts along the way.

Take for example particle physics, where it is hard to complete the calculations in full generality using the present "full-fledged" theory, which furthermore still might miss even more particles. However, in a given energy range it is enough to consider only a couple of particles to obtain relevant quantitative predictions. The existence of other particles only leads to some corrections, which would be desirable to systematically address. This is the main idea of *Effective Field Theories* (EFTs).

Being perturbative in their nature, EFT Lagrangians are essentially infinite series, every next order giving additional corrections. Even if there is a fixed order to which one wants to calculate, there is possibly a lot of operators to consider. It is therefore of interest to find *the operator basis*, which parametrizes all physical phenomena in the realm of the given effective theory, but already does not contain any redundancies.

In general, construction of such basis is immensely difficult because of various non-trivial relations between them. Nonetheless, it is possible to count the number of independent operators of a given type, which certainly helps. The goal of this thesis is to derive the formula for these multiplicities, and to develop all necessary tools along the way.

The group theoretical approach we will develop was pioneered by Henning, Lu, Melia, and Murayama [1]. The research paper is rather long (over 100 pages) and presents multiple approaches, the main one presupposing the knowledge of conformal representations. Therefore, inexperienced reader can have a hard time comprehending details as well as main ideas. We will try to complement this.

We assume the reader is well versed in the subjects of linear algebra and differential geometry. Knowledge of some elementary constructions in group theory, such as cosets and quotient groups, is also assumed. Otherwise, attempt at a mostly self-contained exposition was made. All important theory (for our purposes) about Lie groups and their representations was essentially built up from the beginning, of course together with all necessary proofs.

Quick summary

In **Chapter 1** we briefly introduce the framework of Effective Field Theories, giving us background and physical motivation to study the structure of operators.

In **Chapter 2** we recognize possible relations between operators and define the fundamental object of our interest — *the Hilbert series* — which effectively encodes information about the independent operators of a given type.

In **Chapter 3** we thoroughly build parts of the *representation theory* of compact Lie groups, which shall prove useful for computing the Hilbert series.

In **Chapter 4** we finally derive *the Master Formula* for the Hilbert series, utilizing everything we learned in previous chapters.

In **Chapter 5** we apply the developed formalism to count operators for a single *scalar field*, and also in a more complex case of the *electromagnetic field*. We highlight numerous intricacies, which are present even in these basic theories.

1. Effective Field Theories

Wondering about physics one may stumble upon a fundamental question:

How can we calculate anything without knowing everything?

We do not have the *Theory of Everything*, yet we are able to perform vast amount of precise calculations in practically every domain of the Universe.

The predictive power of physical theories is rooted in the fact, that only a couple of relevant degrees of freedom play an important role in a given problem. The act of neglecting the rest is the first step in progress of understanding any physical phenomena. One could claim that in a certain sense [2]

all physical theories are *effective theories*,

where the word *effective* reflects the restriction of a given theory to only a certain range of effects while being oblivious to the rest.

Just take for example *Newton's mechanics*, which often works only with point masses as idealization of particles with possibly complicated internal structure. Yet, it offers brilliant tools for our everyday life and explanations of many effects, ranging from the usage of a lever to the motion of celestial bodies.

Another example can be found in familiar theories of *electrostatics*, *magnetism* and *optics*, separately being useful subjects without many references between each other, which were only later realized as a manifestation of unified *Maxwell's theory of electromagnetism*.

Our current "fundamental" theories of electroweak and strong interactions are formulated in a language of *quantum field theory* and *gauge theory*. Together with classification of all known elementary particles they are contained in what is called the *Standard Model* of particle physics. Still, we know that the Standard Model can not be complete, as it (among other things) lacks account of the General Relativity. Currently, the Standard Model is often viewed as a low-energy *effective field theory* (EFT) of some more complete theory [3].

1.1 Advantages of EFTs

In the rest of the thesis, we restrict ourselves to *field* theories, that is theories fully described by the action as an integral of the Lagrangian density over the whole space-time manifold. Moreover, we will assume the context of particle physics, where together with *Lorentz transformations* we often encounter additional *gauge transformations*. Naturally, the action must be invariant with respect to both.

To be more precise, the term *effective field theory* in the context of *particle physics* is used in a more strict sense. We will not need most of the following, but to have some background and motivation for our problem we mention some important features, examples, and advantages of EFTs.

Definition 1.1 (Effective field theory)**.** A low-energy *effective field theory* (EFT) $\mathcal{T}$ is described by [2, 4]:

(a) *degrees of freedom/particle fields*, whose behavior $\mathcal{T}$ tries to explain

(b) the *domain of validity*, usually characterized by energy scale Λ. For energies well below characteristic energy scale ($E \ll \Lambda$), $\mathcal{T}$ properly explains physical phenomena of interest. For energies approaching the characteristic energy scale ($E \approx \Lambda$) a transition to a more complete theory or a different and more appropriate EFT may be necessary.

(c) the *perturbation expansion* of action $S_{\mathcal{T}}$ in a suitable parameter (prime example being E/Λ), correcting for effects of initially ignored "irrelevant" degrees of freedom by introducing additional interaction terms.

Remark 1.2. Previous statement about all theories being effective theories can be viewed as talking about leading-order effective theories.

Example 1.3. Here we just mention some well-known EFTs [5]:

- *Fermi theory of weak interactions* — effective theory for the *weak interactions* at energies below the $W^{\pm}$ and Z^0 masses.
- *Heavy Quark Effective Theory* (HQET) — low-energy dynamics of *hadrons* containing a *heavy quark*
- *Chiral Perturbation Theory* (χPT) — dynamics of *pions* in a strongly coupled low-energy limit of *Quantum ChromoDynamics* (QCD)
- *Standard Model Effective Field Theory* (SMEFT) — used to analyze deviations from the *Standard Model* and search for *Beyond Standard Model* physics

Remark 1.4. There are numerous advantages of using EFTs, many being hard to appreciate without a deeper dive. We only list some with a broad relevance [2, 5]:

- even if we are equipped with a more complete theory, calculations in a certain energy range can be drastically simplified by using an appropriate EFT
  - from the beginning we are neglecting aspects unimportant for the problem and remain focused on the relevant degrees of freedom
  - isolation of relevant degrees of freedom may reveal new symmetries that otherwise would have remained obscured
- when dealing with yet unknown physics, EFTs provide us with a systematic perturbative parametrization of physics in a certain energy domain
  - the *main idea* — if we believe our problem can be understood by means and tools of quantum field theory, we can not go wrong by starting with *"the most general Lagrangian"* consistent with our assumptions of locality, unitarity, Lorentz and gauge invariance, ...
  - unified framework utilizing perturbation expansion containing only a few coefficients can be useful for placing model-independent constraints

1.2 Construction of EFTs

The construction of some particular low-energy EFT can be outlined by the following prescription [4]:

(1) Identify all relevant degrees of freedom $\{\Phi_i\}$ and symmetries they satisfy.

(2) Construct the most general effective action consistent with locality, Lorentz invariance and additional (gauge, global, etc.) symmetries,

$$S_{\text{eff}}[\{\Phi_i\}] = \int_{\mathcal{M}} d^{\mathsf{d}}x \left[\mathcal{L}_{\text{kin}}(\{\Phi_i, \partial_a \Phi_i\}) + \sum_j \frac{c_j}{\Lambda^{\Delta_j - \mathsf{d}}} \mathcal{O}_j \right], \qquad (\infty)$$

where d is the dimension of the space-time manifold $\mathcal{M}$, $\mathcal{L}_{\text{kin}}$ is the kinetic part of the Lagrangian density, Λ is the characteristic energetic scale of the EFT and c_j is a dimensionless "Wilson coefficient" corresponding to the local operator $\mathcal{O}_j$ with scaling dimension $\Delta_j \equiv \Delta[\mathcal{O}_j]$.

(3) Calculate physical observables (to the required order of accuracy).

(4) Determine Wilson coefficients c_j by matching the EFT to the already existent underlying model or to the experimental measurements.

In this thesis we will focus on a part of the step **(2)**, which has certain freedom in the choice of the set of operators $\{\mathcal{O}_j\}$. Assuming someone already did step **(1)** for us, our goal will be to count all independent operators figuring in the effective action (∞). Precise meaning of this will be clarified in Chapter 2.

Steps **(3)** and **(4)** are mentioned here only for completeness, nonetheless they are of paramount importance and can be immensely deep.

For simplicity we restrict our attention only to *linearly realized* symmetries, and we will not worry about discrete global symmetries, such as parity, charge conjugation or time inversion. Throughout the thesis we also assume an implicit *Wick rotation* from the Minkowski to the Euclidean space, thus for our *Lorentz group* we can take $\mathsf{SO}(\mathsf{d})$, since we will ignore inversions.

Of course, explicitly constructing the basis of $\{\mathcal{O}_j\}$ would be even better, but it is in general a formidable task. The next best thing is to at least gather some information about quantity of operators of different types, which fortunately turns out to be a feasible goal.

2. The Operator Basis

The notion of operators is probably clear, but before advancing further, it is better to fix the terminology to avoid possible ambiguities.

Definition 2.1 (Operator, invariant/scalar operator)**.** An *operator* $\mathcal{O}$ is a local object composed only from the particle fields and their derivatives at a given point of space-time. An operator is called *invariant* or *scalar* if it is Lorentz and gauge invariant (and also invariant under any other symmetry group). Only scalar operators can appear in the Lagrangian density, since the action can not depend on the choice of an observer.

Remark 2.2. Operators naturally form a linear space, since we are free to multiply them by a constant and add them together. We can thus specify the EFT by the choice of a single scalar *Lagrangian* operator $\mathcal{L}$ from the big space of all operators (actually, it is usually a series in a suitable parameter, see Definition 1.1).

As we suggested before, operators with different appearance can lead to the same physical effect. Equivalently, a mapping from the space of all Lagrangians to the space of S-matrix operators has nonzero kernel. Unsurprisingly, it is advantageous to work with a minimal set of operators covering all possible physical phenomena in a given EFT, thus we come to the following definition.

Definition 2.3 (Operator basis)**.** *The operator basis* $\mathcal{B}$ of the EFT is a minimal set of operators leading to all physical phenomena in the realm of the EFT. We will denote the space of all operators generated by $\mathcal{B}$ as $\mathcal{K} \equiv \operatorname{Span} \mathcal{B}$.

Remark 2.4. As in any vector space, there is a bunch of equivalent operator bases for a given EFT. We will loosely refer to any chosen one as *the operator basis*.

In this chapter we will recognize possible *relations*/redundancies/constraints between operators, which effectively lower the number of linearly independent operators, thus reduce the size of $\mathcal{B}$ and the dimension of $\mathcal{K}$.

Next, we will define a *weighting scheme* for categorizing different types of operators, enabling us to compactly encode reduced information about the operator basis into a generating function called the *Hilbert series*, our prime object of interest. More specifically, it will contain the numbers of independent operators for all distinct types.

2.1 Equivalence relations between operators

From classical theoretical mechanics we know very well, that the form of the Lagrangian is not entirely fixed. By performing a transformation to a different set of *generalized coordinates*, we visually change the Lagrangian, but the physics is the same (we have just changed the parametrization of the configuration space).

Note, that terms proportional to the *equations of motion* are in fact just trivial. Additionally, *total derivative terms* do not change the dynamics, as they reduce to boundary terms in the actions, which do not play a role in a variational principle with fixed endpoints.

But how does the story play out for an effective quantum field theory? Quite similarly, the total derivative (divergence) terms do not contribute to the effective action (∞) (under the usual assumption of vanishing particle fields at infinity). However, in quantum theory we can not eliminate operators proportional to the equations of motion simply by just substituting as in the classical case.

Nonetheless, we are still able to perform *field redefinitions*, which allow us to systematically push operators proportional to the lowest order equations of motion (which are generated from $\mathcal{L}_{\text{kin}}$) to a higher order in the perturbation expansion. A more in-depth explanation is provided by Manohar [5, Chapter 6]. Thus, for the purpose of counting independent operators, we can simply ignore such operators.

Last, but by no means least, we have relations induced by the finite-dimensionality of the space-time manifold. To give a simple example, assume we are in $\mathsf{d} = 2$ and we have one scalar field Φ. Since in $\mathsf{d} = 2$ the antisymmetrization of 3 indices is automatically zero, we obtain the following relation between scalar operators (we first make the antisymmetrization, and after that we contract the indices)

$$\begin{aligned} 0 &\overset{!}{=} \partial^{[a}\partial_a\Phi\partial^b\partial_b\Phi\partial^{c]}\partial_c\Phi \\ &\equiv \partial^a\partial_a\Phi\partial^b\partial_b\Phi\partial^c\partial_c\Phi + \overbrace{\partial^c\partial_a\Phi\partial^a\partial_b\Phi\partial^b\partial_c\Phi + \partial^b\partial_a\Phi\partial^c\partial_b\Phi\partial^a\partial_c\Phi}^{2\partial^b\partial_a\Phi\partial^c\partial_b\Phi\partial^a\partial_c\Phi} \\ &\quad \underbrace{-\partial^b\partial_a\Phi\partial^a\partial_b\Phi\partial^c\partial_c\Phi - \partial^c\partial_a\Phi\partial^b\partial_b\Phi\partial^a\partial_c\Phi - \partial^a\partial_a\Phi\partial^c\partial_b\Phi\partial^b\partial_c\Phi}_{-3\partial_a\partial_b\Phi\partial^a\partial^b\Phi\partial^c\partial_c\Phi} \\ \implies \quad 0 &\overset{!}{=} \overbracket{\partial\partial}\Phi\overbracket{\partial\partial}\Phi\overbracket{\partial\partial}\Phi + 2\overbracket{\partial\partial\Phi\partial\partial\Phi\partial\partial}\Phi - 3\overbracket{\partial\partial\Phi\partial\partial}\Phi\overbracket{\partial\partial}\Phi, \end{aligned}$$

where in the last line we just opted for an alternative notation for contractions.

These conditions are analogous to the statement that $r > \mathsf{d}$ vectors $\{\boldsymbol{v}_i\}_{i=1}^{r>\mathsf{d}}$ can not be linearly independent, thus invariants created from them obey some relations. Assuming we are in Euclidean space, by organizing them in the matrix $A \equiv (\boldsymbol{v}_1|\dots|\boldsymbol{v}_r)$ we can compute the corresponding *Gram matrix*, which is essentially a matrix of all scalar products $\boldsymbol{v}_i \cdot \boldsymbol{v}_j$ ($A^\intercal$ denotes the transpose of A)

$$G \equiv A^\intercal A = \begin{pmatrix} \boldsymbol{v}_1\cdot\boldsymbol{v}_1 & \boldsymbol{v}_1\cdot\boldsymbol{v}_2 & \dots & \boldsymbol{v}_1\cdot\boldsymbol{v}_r \\ \boldsymbol{v}_2\cdot\boldsymbol{v}_1 & \boldsymbol{v}_2\cdot\boldsymbol{v}_2 & \dots & \boldsymbol{v}_2\cdot\boldsymbol{v}_r \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v}_r\cdot\boldsymbol{v}_1 & \boldsymbol{v}_r\cdot\boldsymbol{v}_2 & \dots & \boldsymbol{v}_r\cdot\boldsymbol{v}_r \end{pmatrix}.$$

It is a basic fact from linear algebra that $\operatorname{Ker} A = \operatorname{Ker} A^\intercal A$, from which follows that G is singular if and only if $\{\boldsymbol{v}_i\}$ are linearly dependent. We can apply the same argument for every $(\mathsf{d}+1)\times(\mathsf{d}+1)$ submatrix of G, from which we obtain the so-called *Gram determinant conditions* asserting that every $(\mathsf{d}+1)\times(\mathsf{d}+1)$ minor of G is zero, giving us relations between scalar products of vectors $\{\boldsymbol{v}_i\}$.

Bringing it all together, we conclude our discussion in the following definition.

Definition 2.5 (Operator relations)**.** Operators $\mathcal{O}_m$ and $\mathcal{O}_n$ are considered equivalent (denoted by $\mathcal{O}_m \sim \mathcal{O}_n$), if they satisfy either one of the following relations or possibly their combination [6]:

(a) *Equations of motion* — **EOM** — the usage of field redefinition argument generates relations of the form

$$\mathcal{O}_m \overset{\text{EOM}}{\sim} \mathcal{O}_n \quad \overset{\text{def}}{\iff} \quad \exists\, \mathcal{O}', \Phi_j \colon\ \mathcal{O}_m = \mathcal{O}_n + \frac{\delta S_{\text{kin}}}{\delta \Phi_j} \mathcal{O}'.$$

(b) *Integration by parts* — **IBP** — we also have relations of the type

$$\mathcal{O}_m \overset{\text{IBP}}{\sim} \mathcal{O}_n \quad \overset{\text{def}}{\iff} \quad \exists\, \mathcal{O}' \colon\ \mathcal{O}_m = \mathcal{O}_n + \partial \cdot \mathcal{O}',$$

since $\int_{\mathcal{M}} \partial \cdot \mathcal{O} = 0$, assuming $\mathcal{O}$ vanishes at infinity in the case where the space-time manifold $\mathcal{M}$ is non-compact.

(c) *Gram determinant conditions* — **GDC** — some operators are trivially zero by the finite-dimensionality of $\mathcal{M}$, inducing us relations

$$\mathcal{O}_m \overset{\text{GDC}}{\sim} \mathcal{O}_n \quad \overset{\text{def}}{\iff} \quad \exists\, \mathcal{O}' \Big|_d \begin{cases} = 0 \text{ for } d = \dim \mathcal{M} \\ \neq 0 \text{ for } d \gg \dim \mathcal{M} \end{cases} \colon\ \mathcal{O}_m = \mathcal{O}_n + \mathcal{O}'(d).$$

Remark 2.6 (Kinematic polynomial rings)**.** Another interesting and useful approach to counting and even building the operator basis passes to the momentum space by the Fourier transformation of particle fields, thereby replacing operators with derivatives by *polynomials* in momenta. Operator relations are then represented by relations between polynomials, namely EOM act as *on-shell* conditions, IBP manifest as the statement of *momentum conservation*, and GDC are precisely Gram determinant conditions between scalar products of particle momenta or *Mandelstam variables*. The operator basis is then formed as a certain *quotient ring* with the ideal generated by these relations. It is recommended to at least take a look at the beginning of Section 5 in [1]. However, GDC relations are highly non-trivial to solve using this approach, whereas they will be efficiently accounted for within the group theoretical formalism we are going to develop.

2.2 Generating functions, Hilbert series

Consider an EFT with N degrees of freedom (particle fields) $\{\Phi_i\}_{i=1}^N$. As it is desirable to reduce the complexity of full information about the operator basis, we introduce the concept of *operator weight*, which defines the operator "type".

Definition 2.7 (Weight of an operator)**.** Let $\mathcal{O}$ be an operator of the form $\partial^n \mathbf{\Phi}^{\boldsymbol{r}}$, meaning that it is composed of $\boldsymbol{r} \equiv (r_1, \ldots, r_N)$ powers of $\mathbf{\Phi} \equiv (\Phi_1, \ldots, \Phi_N)$ and n derivatives distributed amongst them. Then we define its *weight* as

$$w[\mathcal{O}] \equiv \boldsymbol{\phi}^{\boldsymbol{r}} \mathcal{D}^n,$$

where we use notation $\boldsymbol{\phi}^{\boldsymbol{r}} \equiv \phi_1^{r_1} \ldots \phi_N^{r_N}$.

Remark 2.8. Weighting scheme in Definition 2.7 retains the most information about the basis while still having the advantage of reducing the full information. Other weightings can be obtained by a change of parametrization, for example $\mathcal{D} \mapsto t$ and $\phi_i \mapsto t^{\Delta[\Phi_i]}$ gives us the *scaling dimension weighting*, $w[\mathcal{O}] \mapsto t^{\Delta[\mathcal{O}]}$.

Now, we would like to have an object encoding information about the basis, which in our case is the number of independent operators with a given weight. Notion of *generating functions* turns out to be exactly what we look for.

Definition 2.9 (Hilbert series)**.** Hilbert series of a given EFT is a formal series in complex parameters $\mathcal{D}$ and $\boldsymbol{\phi} \equiv (\phi_i)_{i=1}^N$ defined by

$$H(\boldsymbol{\phi}, \mathcal{D}) = \sum_{\mathcal{O}\in\mathcal{B}} w[\mathcal{O}] = \sum_{\boldsymbol{r}} \sum_{n=0}^{\infty} d_{\boldsymbol{r}n} \boldsymbol{\phi}^{\boldsymbol{r}} \mathcal{D}^n,$$

where $\mathcal{B}$ is the operator basis and $d_{\boldsymbol{r}n} \equiv d_{r_1\dots r_N n} \in \mathbb{N}_0$ is the number of independent operators in $\mathcal{B}$ with the weight $\boldsymbol{\phi}^{\boldsymbol{r}}\mathcal{D}^n$.

In a certain domain, usually for $|\mathcal{D}|, |\phi| < 1$, the formal series in the definition can be interpreted as a converging series defining a holomorphic function.

To obtain simple expressions for Hilbert series, we also include trivial constant operator with weight 1, which is the only operator with no field content.

Remark 2.10. Hilbert series are usually defined in a context of graded algebras or their generalizations [7]. Nonetheless, we will always refer to Hilbert series in the spirit of Definition 2.9.

Remark 2.11. Power of the Hilbert series (or generating functions in general) arises in the cases, where it can be expressed in a closed form. This can be understood either in the context of formal power series, or (in domain of convergence) as really the function obtained by summation of the convergent series.

Remark 2.12. As will be shown several times, a typical summed up Hilbert series takes the form of

$$H = \frac{N}{D}, \quad \text{with} \quad D \equiv \prod_{\mathcal{O}\in\mathcal{G}} (1 - w[\mathcal{O}]),$$

where $\mathcal{G}$ is the set of *generators*, which can be multiplied repeatedly to form operators of higher weight. Form of the denominator D reflects this fact, which be can seen by expanding it as a geometric series. The numerator N encodes additional operators (which upon repetition decompose to operators generated from $\mathcal{G}$) and possible relations between operators. In the case of a *freely generated* basis, the numerator is unity and all operators can be generated from $\mathcal{G}$.

Remark 2.13. Sometimes it is useful to write the Hilbert series as

$$H(\boldsymbol{\phi}, \mathcal{D}) = \sum_{\boldsymbol{r}} \boldsymbol{\phi}^{\boldsymbol{r}} H_{\boldsymbol{r}}(\mathcal{D}), \quad \text{where} \quad H_{\boldsymbol{r}}(\mathcal{D}) \equiv \sum_{n=0}^{\infty} d_{\boldsymbol{r}n} \mathcal{D}^n.$$

We will refer to $H_{\boldsymbol{r}}(\mathcal{D})$ also by the term Hilbert series, where we are just considering operators with a fixed field content, that is of the form $\partial^n \boldsymbol{\Phi}^{\boldsymbol{r}}$ for a given $\boldsymbol{r}$.

Example 2.14. Some examples of the Hilbert series are obtained by considering the theory of N scalar fields in $\mathsf{d} = 1$ with the kinetic term

$$\mathcal{L}_{\text{kin}}(\{\Phi_i, \partial\Phi_i\}) \equiv -\sum_{i=1}^{N} \frac{1}{2}(\partial\Phi_i)^2.$$

It is particularly simple, because in the absence of the indices there are no Gram determinant conditions, and every operator is automatically invariant. Nevertheless, it will be very instructive. For a comprehensive analysis take a look at the work of Henning, Lu, Melia, and Murayama [6]. We will go through some of the corner cases:

(1) *No relations.* This case is trivial, because the operator basis can be easily guessed, namely it is freely generated by the set $\{\partial^n\Phi_i\}$ with $i = 1, \ldots, N$ and $n \in \mathbb{N}_0$. In other words, every operator is obtained in the expansion of

$$\prod_{i=1}^{N}(1 + \Phi_i + \Phi_i^2 + \ldots)(1 + \partial\Phi_i + (\partial\Phi_i)^2 + \ldots)(1 + \partial^2\Phi_i + (\partial^2\Phi_i)^2 + \ldots)\cdots$$
$$= \prod_{i=1}^{N} \frac{1}{(1-\Phi_i)(1-\partial\Phi_i)(1-\partial^2\Phi_i)\cdots} = \prod_{i=1}^{N}\prod_{n=0}^{\infty} \frac{1}{1-\partial^n\Phi_i}.$$

Corresponding *free Hilbert series* is obtained by substituting (Φ_i, ∂) for $(\phi_i, \mathcal{D})$,

$$H_N^{\text{free}}(\boldsymbol{\phi}, \mathcal{D}) = \prod_{i=1}^{N}\prod_{n=0}^{\infty} \frac{1}{1-\mathcal{D}^n\phi_i}.$$

(2) *Only EOM relations.* Considering the kinetic Lagrangian density $\mathcal{L}_{\text{kin}}$, the relations generated by free equations of motion are $\partial^2\Phi_i = 0$. Even simpler than before, because we have $\partial^2\Phi_i = 0 \Rightarrow \partial^n\Phi_i = 0$ for $n \geq 2$, the operator basis is now freely and finitely generated by the set $\{\Phi_i, \partial\Phi_i\}_{i=1}^N$, giving us the *EOM Hilbert series* of the form

$$H_N^{\text{EOM}}(\boldsymbol{\phi}, \mathcal{D}) = \prod_{i=1}^{N} \frac{1}{(1-\phi_i)(1-\mathcal{D}\phi_i)}.$$

(3) *Only IBP relations.* Relations between $d_{\boldsymbol{r}n,\text{free}}$ operators of the form $\partial^n\boldsymbol{\Phi}^{\boldsymbol{r}}$ are formed from $d_{\boldsymbol{r}n-1,\text{free}}$ operators of the form $\partial^{n-1}\boldsymbol{\Phi}^{\boldsymbol{r}}$ (with one less derivative) by performing a total derivative. One particular example in the case of only one field flavor would be

$$0 \sim \partial\Big((\partial^{n-1}\Phi)\Phi^{r-1}\Big) = (\partial^n\Phi)\Phi^{r-1} + (r-1)(\partial^{n-1}\Phi)(\partial\Phi)\Phi^{r-2}.$$

Due to linearity of the derivative, we obtain a linearly independent set of relations from a linearly independent set of operators, with only exception being the trivial constant operator, which does not generate any condition. Therefore, we have (δ_{ij} denotes the Kronecker delta)

$$d_{\boldsymbol{r}n}^{\text{IBP}} = \begin{cases} d_{\boldsymbol{r}n}^{\text{free}} - d_{\boldsymbol{r}n-1}^{\text{free}} & \boldsymbol{r} \neq \boldsymbol{0} \\ \delta_{0n} \equiv d_{\boldsymbol{0}n}^{\text{free}} & \boldsymbol{r} = \boldsymbol{0} \end{cases}.$$

Simple reordering of the summation gives us

$$H_N^{\mathrm{IBP}}(\boldsymbol{\phi},\mathcal{D}) = \sum_{\boldsymbol{r}}\sum_{n=0}^{\infty} d^{\mathrm{IBP}}_{\boldsymbol{r}n}\boldsymbol{\phi}^{\boldsymbol{r}}\mathcal{D}^n = 1 + \sum_{\boldsymbol{r}\neq\boldsymbol{0}}\sum_{n=0}^{\infty}(d^{\mathrm{free}}_{\boldsymbol{r}n} - d^{\mathrm{free}}_{\boldsymbol{r}n-1})\boldsymbol{\phi}^{\boldsymbol{r}}\mathcal{D}^n$$
$$= 1 + (1-\mathcal{D})\sum_{\boldsymbol{r}\neq\boldsymbol{0}}\sum_{n=0}^{\infty} d^{\mathrm{free}}_{\boldsymbol{r}n}\boldsymbol{\phi}^{\boldsymbol{r}}\mathcal{D}^n = \mathcal{D} + (1-\mathcal{D})H_N^{\mathrm{free}}(\mathcal{D},\{\phi_i\}).$$

Again, the *IBP Hilbert series* is given by a straightforward modification of the *free Hilbert series* as

$$H_N^{\mathrm{IBP}}(\boldsymbol{\phi},\mathcal{D}) = \mathcal{D} + (1-\mathcal{D})\prod_{i=1}^{N}\prod_{n=0}^{\infty}\frac{1}{1-\mathcal{D}^n\phi_i}.$$

(4) *Both EOM and IBP relations.* Interplay of both types of relations makes this problem harder (and therefore interesting), but it is still possible to find analytic expression using techniques similar to those we will present in following chapters [6]. We do not have the space to cover it here, but it is advised for the reader to take a look at its solution (mainly Section 4 of [6]), probably best as complement to our Chapter 4.

In fact, this was really meant as a training toy model for exploring the simplest possible but already non-trivial computation of operator basis including both EOM and IBP relations.

A natural question arises:

How do we obtain the Hilbert series in a more general setting, for arbitrary space-time dimension and field content with possible internal symmetries?

So far we have only encountered the case with $\mathsf{d} = 1$ and no internal symmetries. Only in this dimension the Lorentz group is trivial and an application of derivative is always unambiguous. From now on, we always assume $\mathsf{d} \geq 2$, where each derivative carries an index representing non-trivial transformation properties. To construct a Lorentz invariant operator, we are forced to somehow contract all the indices. Similar thing goes for internal symmetries, both additions bringing a *substantial complexity*. It suffices to say that it is far from obvious how to generalize the discussion in 2.14.

Nonetheless, the toy model in 2.14 has proved to be full of inspiring lessons, mainly the case **(4)**, paving the way for vast generalization developed by the same authors — Henning, Lu, Melia, and Murayama [1]. In the next chapter, we will start the preparatory work essential for systematic treatment of our problem.

3. Representations of Lie Groups

Groups are omnipresent in the realm of mathematics and physics, and so are *manifolds* in the context of field theories. It is impossible to go even past the proper formulation of the physical theory without specifying *transformation properties* of the present particle fields. Since everything plays out on a smooth manifold, the corresponding symmetry groups are themselves often smooth manifolds.

Groups and their *representations* will precisely turn out to be the tools we need to systematically count the Hilbert series. Idea is that by continuously being aware of the transformation properties — representations — of different operators, we can automatically rule out operators that generate GDC conditions, and also project out the number of only invariant operators. Furthermore, we will learn to project out the number of operators corresponding to any representation, which will be crucial for adressing IBP relations.

In **Section 3.1** we will introduce the notion of *Lie groups* as well as particular examples we are interested in. Moreover, by studying their differential structure, we will be able to show unique existence of the *invariant Haar integral* on *compact* Lie groups, on which much of the following results build upon.

In **Section 3.2** we will start the study of group *representations*. By borrowing concepts from linear algebra we will be able to obtain wide range of representations describing various transformation properties of different objects.

In **Section 3.3** we will dive deeper in the representation theory with the goal to systematically examine the structure of representations. We will introduce the notion of *irreducible* representations and show that any finite-dimensional representation of a compact Lie group *decomposes* to such atomic pieces.

In **Section 3.4** we will finish our endeavor to find the *multiplicities* of irreducible representations in any given representation by deriving the *projection formula*. It works by assigning *characters* to corresponding representations and integrating over the group, utilizing the fact that *irreducible* characters turn out to be orthonormal.

In **Section 3.5** we will extend our knowledge to the *graded representations*, which enable us to work with countable number of representations in parallel, since they are in direct relation with generating functions.

In **Section 3.6** we will once again return to direct study of compact Lie groups. We will introduce numerous important concepts, such as *maximal torus* and *roots* of a group, ultimately leading to the *Weyl integration formula*, which provides us with explicit means of performing the integral in the projection formula.

We only introduce important notions from group and representation theory, which shall prove useful for our problem. This chapter is mainly based on the beautiful works of Bröcker and Dieck [8], Fulton and Harris [9], and Sepanski [10].

3.1 Compact Lie groups

Lie groups are essentially a harmonious combination of the algebraic structure of a group and the analytic and geometric structure of a differentiable manifold. They underlay the fundamental theory describing the continuous transformations.

Definition 3.1 (Lie group)**.** A *Lie group* G is a *group* that is also a *differentiable manifold*, for which the group multiplication $m\colon G \times G \to G$ is a *smooth* map.

Remark 3.2. It can be shown that the definition of a Lie group already implies also the smoothness of the inverse map $\iota : G \to G, g \mapsto g^{-1}$ [8].

Remark 3.3. We will use e or $\mathbf{1}$ to denote the identity of a group G. For ease of notation we write just gh instead of $m(g; h)$ for any $g, h \in G$.

Some Lie groups can have *similar structures*. To get a grasp at what it means, we introduce the elementary notion of *homomorphisms*.

Definition 3.4 (Homomorphism of Lie groups)**.** A *homomorphism of Lie groups* is a smooth map $\phi\colon G \to H$ between two Lie groups G and H, such that the *group operation is preserved*, that is

$$\phi(gh) = \phi(g)\phi(h) \quad \text{for all } g, h \in G,$$

where the multiplication on the left side is in G and on the right side in H.

Remark 3.5. There always exists a *trivial homomorphism*, which maps every element of G to the identity element of H. If homomorphism between groups is non-trivial, it is a sign they share certain similarities. Bijective homomorphism is called *isomorphism*, corresponding groups are then called *isomorphic* ($G \cong H$), which means they are identical up to differences in the notation of their elements.

Example 3.6. Every finite-dimensional *vector space* V with its additive group structure is a Lie group. By any choice of basis it is isomorphic to $\mathbb{R}^{\dim V}$.

We can construct new groups by a simple combination of already known ones.

Definition 3.7 (Direct product of Lie groups)**.** Let G and H be Lie groups. Then their *direct product* $\mathsf{G} \equiv G \times H$ is also a Lie group, where we understand it both as a direct product of their manifold structures and their group structures. Namely, the group multiplication of $\mathsf{g}_1, \mathsf{g}_2 \in \mathsf{G}$ is defined component-wise via

$$\mathsf{g}_1\mathsf{g}_2 \equiv (g_1; h_1)(g_2; h_2) \equiv (g_1 g_2; h_1 h_2), \quad \text{where } g_1, g_2 \in G \text{ and } h_1, h_2 \in H.$$

Example 3.8. The *torus* $\mathbb{T}^n$ ($n \in \mathbb{N}$) defined by

$$\mathbb{T}^n \equiv \mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n \cong (S^1)^n \equiv \overbrace{S^1 \times \cdots \times S^1}^{n}$$

is a Lie group, where the circle group $S^1 \equiv \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in $\mathbb{C}$ viewed as a multiplicative subgroup of $\mathbb{C}^\times \equiv \mathbb{C} \setminus \{0\}$. The isomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$ is given by the map $\mathbb{R}/\mathbb{Z} \to S^1 : t \mapsto e^{2\pi i t}$.

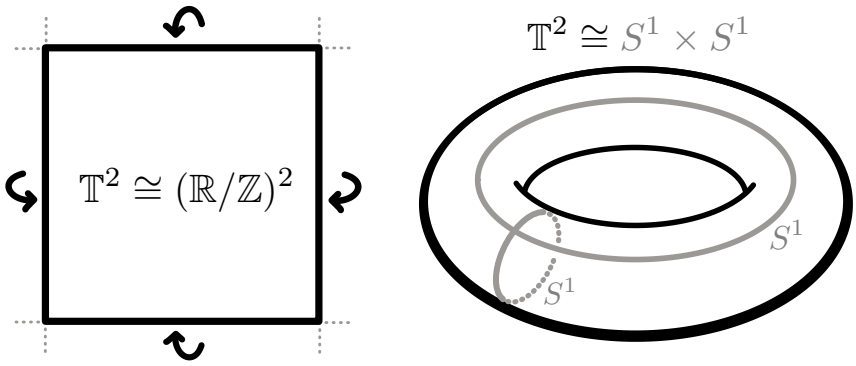


Figure 3.1: Illustration of alternative isomorphic definitions of the torus $\mathbb{T}^2$.

Example 3.9 (General linear groups)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. By $\mathrm{End}(V)$ we denote the space of *endomorphisms* of V, that is the vector space of linear maps from V to itself. Very important is the set $\mathsf{GL}(V) \subset \mathrm{End}(V)$ of *automorphisms* (invertible endomorphisms) of V. Equivalently, we have

$$\mathsf{GL}(V) = \{A \in \mathrm{End}(V) \mid \det A \neq 0\}.$$

Determinant is a continuous function, thus $\mathsf{GL}(V)$ is an open subset of $\mathrm{End}(V)$. It is easy to see that $\mathsf{GL}(V)$ has the structure of a differentiable manifold and in coordinates the group operation is the matrix multiplication, which is smooth. Therefore $\mathsf{GL}(V)$ is a Lie group, particularly we have the *General Linear* groups

$$\mathsf{GL}(n, \mathbb{R}) \equiv \mathsf{GL}_{\mathbb{R}}(\mathbb{R}^n) \quad \text{and} \quad \mathsf{GL}(n, \mathbb{C}) \equiv \mathsf{GL}_{\mathbb{C}}(\mathbb{C}^n),$$

which are canonically isomorphic to the groups of invertible $n \times n$ matrices over $\mathbb{R}$ and $\mathbb{C}$, respectively. Thus, we can think of $\mathsf{GL}(n, \mathbb{R})$, $\mathsf{GL}(n, \mathbb{C})$, and their subgroups (see Example 3.10) as *matrix groups* with the operation of matrix multiplication.

Example 3.10 (Special Orthogonal and Unitary groups)**.** We will be mostly interested in compact Lie groups, which will represent our Lorentz and gauge groups. It is a standard textbook result, that a closed subgroup of a Lie group inherits the Lie group structure [8]. We therefore obtain the following *compact classical groups*:

(a) the *Orthogonal* $\mathsf{O}(n)$ and the *Special Orthogonal* $\mathsf{SO}(n)$ groups defined by

$$\mathsf{O}(n) \equiv \{g \in \mathsf{GL}(n, \mathbb{R}) \mid g^\intercal g = \mathbf{1}\}, \quad \mathsf{SO}(n) \equiv \{g \in \mathsf{O}(n) \mid \det g = 1\},$$

where $g^\intercal$ denotes the transpose of g. Group $\mathsf{O}(n)$ splits into two connected components with the values ± 1 of the determinant, one being $\mathsf{SO}(n)$.

(b) the *Unitary* $\mathsf{U}(n)$ and the *Special Unitary* $\mathsf{SU}(n)$ groups defined by

$$\mathsf{U}(n) \equiv \left\{g \in \mathsf{GL}(n, \mathbb{C}) \;\middle|\; g^+ g = \mathbf{1}\right\}, \quad \mathsf{SU}(n) \equiv \{g \in \mathsf{U}(n) \mid \det g = 1\},$$

where $g^+ \equiv \overline{g^\intercal}$ denotes the conjugate transpose of g. It can be shown that both $\mathsf{U}(n)$ and $\mathsf{SU}(n)$ are connected.

These groups preserve the standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$, particularly the norms. Therefore it can be seen they are compact, since they are closed and bounded in the finite-dimensional vector spaces $\mathrm{End}(\mathbb{R}^n)$ and $\mathrm{End}(\mathbb{C}^n)$.

Remark 3.11 (Symplectic groups)**.** There is one other compact Lie group family, namely the *symplectic* groups

$$\mathsf{Sp}(n) \equiv \{g \in \mathsf{GL}(n, \mathbb{H}) \mid g^* g = \mathbf{1}\},$$

where $\mathbb{H}$ denotes the *quaternions* and g^* denotes the quaternionic conjugate transpose of g. Since understanding a noncommutative division algebra $\mathbb{H}$ takes some work and we will not need the symplectic groups, we refer the interested reader to practically any book on the subject of Lie groups [8–10].

Remark 3.12. More generally, classical groups are defined as automorphism groups that preserve a *bilinear* or *sesquilinear* form on finite-dimensional vector spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. For example, $\mathsf{O}(n)$ is defined as a subgroup of $\mathsf{GL}(n, \mathbb{R})$ preserving some symmetric positive-definite bilinear form $Q\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, that is

$$Q(g\boldsymbol{v}, g\boldsymbol{w}) = Q(\boldsymbol{v}, \boldsymbol{w}) \quad \text{for any } g \in \mathsf{O}(n) \text{ and } \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n.$$

Since we can always bring such form Q by a change of basis to the standard scalar product on $\mathbb{R}^n$, we precisely obtain the definition in **(a)** of Example 3.10.

What can we reveal about the Lie groups (namely the compact ones) by studying the interplay of their algebraic and analytic structures? Perhaps not so surprisingly, many strong results are just waiting to be uncovered. We will start by introducing the tensor fields which are adjusted to the group structure of a Lie group.

Definition 3.13 (Left translations, left-invariant fields)**.** Let G be a Lie group. The *left translation* by $g \in G$ is a diffeomorphism $L_g\colon G \to G$ defined by $L_g h \equiv gh$ for any $h \in G$. The inverse is given by $L_g^{-1} = L_{g^{-1}}$.

A tensor field (section of the tensor bundle) on G is called *left-invariant* if it is invariant under the induced action of left translations. That is, a left invariant tensor field $\boldsymbol{A} \in \operatorname{Sect} \boldsymbol{T}^p_q G$ satisfies

$$L_{g*}\boldsymbol{A} = \boldsymbol{A} \quad \text{for every } g \in G,$$

where L_{g*} denotes the pushforward by L_g.

Remark 3.14. We implicitely used the fact that L_g is diffeomorphism, thus we can pushforward whole vector field to obtain another vector field. We also extend the definition of pushforward to forms by $L_{g*} \equiv (L_g^*)^{-1} = L^*_{g^{-1}}$.

Remark 3.15. Analogously we can define the *right translations* by $R_g h \equiv hg$ for any $g, h \in G$, and the corresponding right-invariant tensor fields.

It follows that left-invariant fields are uniquely given by their value at the identity.

Proposition 3.16 (Left-invariant lift)**.** The tensor space $\boldsymbol{T}^{\,p}_{e\,q}G$ at the identity of a Lie group G is *canonically isomorphic* to the space of the left-invariant tensor fields $\operatorname{Sect}_L \boldsymbol{T}^{\,p}_{\,q}G \subset \operatorname{Sect} \boldsymbol{T}^{\,p}_{\,q}G$ on G. We denote this isomorphism by

$$\begin{aligned} \boldsymbol{\ell}_\bullet \colon \boldsymbol{T}^{\,p}_{e\,q}G &\to \operatorname{Sect}_L \boldsymbol{T}^{\,p}_{\,q}G \\ \boldsymbol{a} &\mapsto \boldsymbol{A} \equiv \boldsymbol{\ell}_{\boldsymbol{a}}, \end{aligned}$$

and it is defined by $\boldsymbol{\ell}_{\boldsymbol{a}}|_g \equiv L_{g*}\boldsymbol{a}$.

Proof. To any left-invariant tensor field $\boldsymbol{A}$ we can assign uniquely a tensor $\boldsymbol{A}|_e$ at the identity, which follows from (we use the left-invariance of $\boldsymbol{A}$)

$$\boldsymbol{A}|_g = (L_{g*}\boldsymbol{A})|_g \equiv L_{g*}(\boldsymbol{A}|_e) \equiv \boldsymbol{\ell}_{\boldsymbol{A}|_e}|_g \quad \Longrightarrow \quad \boldsymbol{A} = \boldsymbol{\ell}_{\boldsymbol{A}|_e}.$$

The map $\boldsymbol{A}|_e \overset{\boldsymbol{\ell}}{\mapsto} \boldsymbol{A}$ is evidently one-to-one and linear, thus isomorphism. □

Remark 3.17. Since the space of top-dimensional forms $\boldsymbol{\Lambda}^{\mathsf{d}}_e G$ at the identity of G is one-dimensional, and pushforward of a nonzero form by a diffeomorphism L_g leads to a nonzero form, we obtain a unique (up to multiplication by a constant) left-invariant *volume form* on G. Specially, every Lie group is orientable.

Now we restrict ourselves to the case of a compact Lie group G, where it is possible to perform integration over the whole G without any problem. We will show there actually exists a unique normalized *invariant Haar integral*.

Theorem 3.18 (Invariant Haar integral)**.** Let G be a compact Lie group and $h \in G$ be any element of G. Denote by $C(G)$ the vector space of continuous real-valued functions on G. Then there exists the *invariant Haar integral*

$$\begin{aligned} \int \colon C(G) &\to \mathbb{R} \\ f &\mapsto \int f \equiv \int_G f(g)\, dg\,, \end{aligned}$$

and is uniquely determined by the following properties:

(a) It is linear, monotone, and *normalized*, that is $\displaystyle\int_G dg = 1$.

(b) It is *left invariant*, that is $\displaystyle\int_G f(hg)\, dg \equiv \int_G f \circ L_h(g)\, dg = \int_G f(g)\, dg$.

Proof. Existence follows from Remark 3.17, giving us a volume form $\boldsymbol{\omega}$ on G, with respect to which we define the integral. Property **(a)** is then satisfied by properly rescaling the $\boldsymbol{\omega}$. Property **(b)** follows from the left-invariance of $\boldsymbol{\omega}$ by

$$\int_G f\circ L_h\, dg \equiv \int_G (f\circ L_h)\boldsymbol{\omega} = \int_G (f\circ L_h) L_h^*\boldsymbol{\omega} = \int_G L_h^*(f\boldsymbol{\omega}) = \int_{L_h G \equiv G} f\boldsymbol{\omega} \equiv \int_G f\, dg\,,$$

where we used that L_h is orientation preserving (since L_h^* preserves $\boldsymbol{\omega}$).

We will prove uniqueness by comparing all left-invariant integrals with a single right-invariant integral (can be constructed similarly as the left-invariant one). Let $f \mapsto \int_G f(g)\,dg$ be any integral with the properties from the definition, and $f \mapsto \int_G f(g)\delta g$ is a given right-invariant normalized integral, that is

$$\int_G \delta g = 1 \quad \text{and} \quad \int_G f(gh)\delta g = \int_G f(g)\delta g.$$

Then for any $f \in C(G)$ by use of left-invariance/right-invariance, normalization, and *Fubini's theorem* to exchange the order of integration, we obtain

$$\begin{aligned}\int_G f(g)\,dg &= \int_G \left(\int_G f(\ \ g)\,dg\right)\delta h = \int_G \left(\int_G f(hg)\,dg\right)\delta h \\ &= \int_G \left(\int_G f(hg)\delta h\right)dg = \int_G \left(\int_G f(h\ \ )\delta h\right)dg = \int_G f(h)\delta h.\end{aligned}$$

This is sufficient to establish uniqueness. □

Corollary 3.19 (Properies of invariant Haar integral)**.** Let G be a compact Lie group, $h \in G$ be any element of G, and ϕ be any automorphism $\phi : G \to G$ (homomorphism from G to itself). Then the *invariant Haar integral* satisfies

$$\int_G f(g)\,dg \overset{(1)}{=} \int_G f(hg)\,dg \overset{(2)}{=} \int_G f(gh)\,dg \overset{(3)}{=} \int_G f(g^{-1})\,dg \overset{(4)}{=} \int_G f\circ\phi(g)\,dg\,.$$

Proof. Suppose $\varphi : G \to G$ is a diffeomorphism, then the map $f \mapsto \int_G f\circ\varphi(g)\,dg$ is clearly a normalized integral. If it happens to also be left-invariant, that is $\int_G f\circ L_h\circ\varphi(g)\,dg = \int_G f\circ\varphi(g)\,dg$, from the uniqueness (see Theorem 3.18) it must be that $\int_G f\circ\varphi(g)\,dg = \int_G f(g)\,dg$. We apply this idea **(⋆)** to prove:

(1) Follows directly from the definition of the invariant Haar integral.

(2) Choose $\varphi(g) \equiv R_h(g) \equiv gh$ and any $k \in G$. Left-invariance follows from

$$\int_G f\circ L_k\circ R_h(g)\,dg = \int_G f(kgh)\,dg = \int_G f\circ R_h(kg)\,dg \overset{(1)}{=} \int_G f\circ R_h(g)\,dg\,.$$

(3) Choose $\varphi(g) \equiv \iota(g) \equiv g^{-1}$. We can use already proved **(2)** to show

$$\int_G f\circ L_h\circ\iota(g)\,dg = \int_G f(hg^{-1})\,dg = \int_G f\circ\iota(gh^{-1})\,dg \overset{(2)}{=} \int_G f\circ\iota(g)\,dg\,.$$

(4) Choose $\varphi(g) \equiv \phi(g)$. By homomorphism property (see Definition 3.4) we have

$$\int_G f\circ L_h\circ\phi(g)\,dg = \int_G f(h\phi(g))\,dg = \int_G f\circ\phi(\phi^{-1}(h)g)\,dg \overset{(1)}{=} \int_G f\circ\phi(g)\,dg\,.$$

Application of the idea **(⋆)** finishes the proof. □

Remark 3.20 (Haar measure)**.** We sometimes refer to dg figuring in the invariant Haar integral as the (invariant) *Haar measure*.

Remark 3.21. Every *finite group* is actually a zero-dimensional compact Lie group (with discrete topology). Everything what we will present about representations applies to finite groups with the appropriate transition from integrals to sums. Historically, it was knowledge about finite groups that was extended to the special case of compact Lie groups by existence of the Haar measure.

The importance of the invariant Haar integral on compact Lie groups can not be understated. We will fundamentally use it in Sections 3.3 and 3.4 to obtain invariant objects by averaging over the group.

For now, we leave the direct analysis of the group structure, as we will approach them somewhat indirectly through their *representations*. We will return once more to their differential geometry in Section 3.6.

3.2 Representations

Groups themselves can be pretty abstract entities. However, their properties can be probed by examining their action on certain simple structures we understand very well, in our case *vector spaces*. This is the subject of *representation theory*, which in a *group structure preserving* way assigns to every group element a linear transformation of a vector space. In essence, it is able to reduce some problems in abstract algebra to problems in linear algebra.

Definition 3.22 (Representation)**.** A *representation* of a Lie group G on a finite-dimensional complex vector space V is a *Lie group homomorphism* $\rho\colon G \to \mathsf{GL}(V)$. The *dimension of the representation* V is $\dim V \equiv \dim_{\mathbb{C}} V$.

Remark 3.23. When there is little ambiguity about the map ρ, we call V itself a representation of G. In that case we say that g has an *action* on V. We often write $g|_V$ for $\rho(g)$ and directly $g\boldsymbol{v}$ for $\rho(g)(\boldsymbol{v})$, where $\boldsymbol{v} \in V$. For a visual representation of the notion of group representations see Figure 3.2.

Remark 3.24. We could extend the definition to infinite-dimensional vector spaces or to vector spaces over different fields $\mathbb{F}$, for example $\mathbb{R}$. Since we will not find much use for such generalizations, and sometimes we will distinctly use finite-dimensionality and properties of complex numbers, we assume finite-dimensional representations over $\mathbb{C}$ unless stated differently.

Example 3.25. Some elementary representations appear automatically:

(a) *Trivial representation* — every group element is represented by an identity operator on any given vector space V. Although this does not seem exciting at the first sight, we are frequently interested in objects which are invariant under the action of the group.

(b) *Standard representation* — the *matrix* groups $\mathsf{GL}(n,\mathbb{R})$, $\mathsf{GL}(n,\mathbb{C})$, and their subgroups (see Examples 3.9 and 3.10) are canonically represented by matrices on the corresponding complex vector space $\mathbb{C}^n$ (we understand $\mathbb{R} \subset \mathbb{C}$).

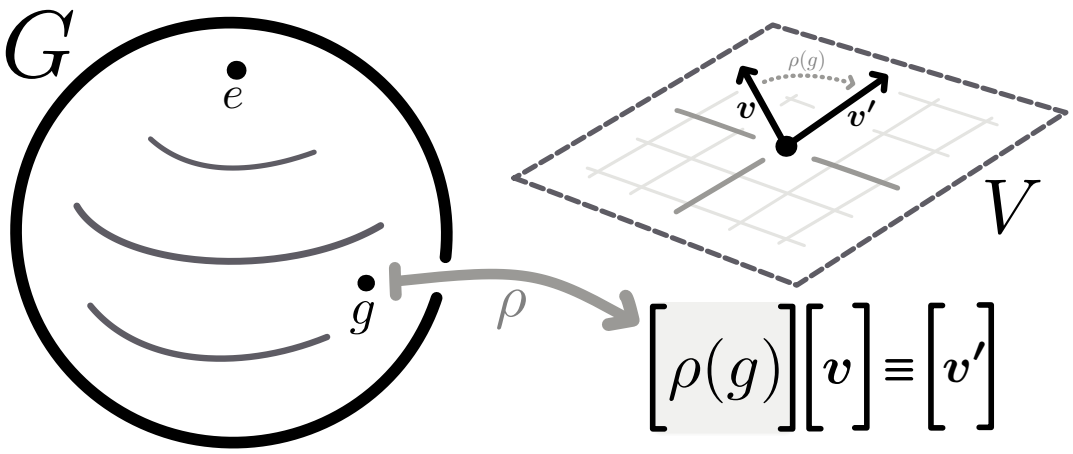


Figure 3.2: Illustration of a representation ρ of group G on vector space V. A group element $g \in G$ is represented by a linear operator $\rho(g) \in \mathsf{GL}(V)$, in particular transforming a vector $\boldsymbol{v} \in V$ to the vector $\boldsymbol{v}' \equiv \rho(g)(\boldsymbol{v}) \in V$.

To enlarge our collection of representations, we will borrow some concepts from linear algebra, which allow us to build more elaborate representations out of known simpler ones. Typical and the most basic constructions are the direct sum and the tensor product of vector spaces.

Definition 3.26 (Direct sum and tensor product of representations)**.** Let V and W be representations of a group G. Then the *direct sum* $V \oplus W$ and the *tensor product* $V \otimes W$ are representations with the actions of any $g \in G$ defined through relations

$$g(\boldsymbol{v} \oplus \boldsymbol{w}) \equiv g\boldsymbol{v} \oplus g\boldsymbol{w}, \quad g(\boldsymbol{v} \otimes \boldsymbol{w}) \equiv g\boldsymbol{v} \otimes g\boldsymbol{w},$$

for any $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$.

We can also induce representation of V on the dual space $V^* \equiv \mathrm{Hom}(V, \mathbb{C})$ of linear maps on V by respecting the duality $\langle \boldsymbol{\alpha}, \boldsymbol{v} \rangle \equiv \boldsymbol{\alpha}(\boldsymbol{v})$, that is by requiring $\langle g\boldsymbol{\alpha}, g\boldsymbol{v} \rangle \equiv \langle \rho^*(g)(\boldsymbol{\alpha}), \rho(g)(\boldsymbol{v}) \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{v} \rangle$ for all $g \in G, \boldsymbol{\alpha} \in V^*$ and $\boldsymbol{v} \in V$.

Definition 3.27 (Dual representation)**.** Let V be a representation of a group G. Then the *dual representation* $\rho^* \colon G \to \mathsf{GL}(V^*)$ is defined by

$$\rho^*(g) = \rho(g^{-1})^\intercal \quad \text{for any } g \in G.$$

Now that we have already defined the dual representation and the tensor product of representations, we can similarly realize representation on $\mathrm{Hom}(V, W)$, the space of linear maps between vector spaces V and W (which themselves are representations), motivated by the canonical isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$ for finite-dimensional V and W.

Definition 3.28 ($\mathrm{Hom}(V,W)$ representation)**.** Let V and W be representations of a group G. Then $\mathrm{Hom}(V,W) \cong V^* \otimes W$ is a representation with the action of any $g \in G$ defined through relation

$$(g\varphi)(\boldsymbol{v}) \equiv g\varphi(g^{-1}\boldsymbol{v}) \quad \text{for any } \varphi \in \mathrm{Hom}(V,W), \boldsymbol{v} \in V.$$

It is often interesting to study maps between representations, which preserve their structure (in similar sense as homomorphisms preserve group structure).

Definition 3.29 (G-map, $\mathrm{Hom}_G(V,W)$, equivalent representations)**.** A *G-map* (*G-linear* map) is a linear map $\varphi\colon V \to W$ between representations V and W of a group G, such that it preserves the group representation structure, that is

$$\varphi(g\boldsymbol{v}) = g\varphi(\boldsymbol{v}) \quad \text{for all } g \in G, \boldsymbol{v} \in V.$$

We denote by $\mathrm{Hom}_G(V,W)$ the space of all G-linear maps between V and W.

Representations V and W are *equivalent* ($V \cong W$) if there exists a bijective G-map between them, which means they are identical up to a change of basis.

Remark 3.30. Comparing Definitions 3.28 and 3.29 we observe, that $\mathrm{Hom}_G(V,W)$ is just a subspace of $\mathrm{Hom}(V,W)$ that transforms trivially under action of G.

Remark 3.31. Previous definitions are equivalent to the statement, that following diagrams are commutative for every $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & \mathbb{C} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathbf{1}} \\ V & \xrightarrow[\varphi]{} & \mathbb{C} \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[g\varphi]{} & W \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

(a) $V^* \equiv \mathrm{Hom}(V,\mathbb{C})$ **(b)** $\mathrm{Hom}(V,W)$ **(c)** $\mathrm{Hom}_G(V,W)$

Figure 3.3: Commutative diagrams corresponding to Definitions 3.27, 3.28 and 3.29.

3.3 Complete reducibility and Schur's Lemma

We have already encountered numerous possibilities, how we can construct other representations from already known ones, most simply by taking the direct sum. Steps in the other direction would be similarly, if not more, of great interest. If we were able to systematically decompose any representation to its smallest pieces, understanding just them would be enough, leading to a major simplification.

In order to decompose a given representation V, there necessarily must exist some proper subspace of V that is somehow closed under the action of a group. We can create another representation by restricting the group on such a subspace.

Definition 3.32 (Subrepresentation)**.** A vector subspace W of a representation V of a group G is called *G-invariant* or a *subrepresentation* if $g\boldsymbol{w} \in W$ for every $g \in G$ and $\boldsymbol{w} \in W$. Thus, $W \leq V$ itself is a representation of G.

Now we define "atomic" representations, in which the action of the group already mixes up the whole space, therefore they can not be decomposed any further.

Definition 3.33 (Irreducible representation)**.** A nonzero representation V is called *irreducible* if the only subrepresentations are the zero space $\{\mathbf{0}\}$ and the whole V. Otherwise, V is called *reducible*.

It would be ideal if any representation could be decomposed into irreducible representations. In the case of general Lie groups, this is not the case. However, as we will show later, representations of compact Lie groups satisfy this property.

Definition 3.34 (Complete reducibility)**.** A representation V of a group G is called *completely reducible* if it is a direct sum of irreducible representations.

But first, to have a glimpse at how useful is to work with irreducible representations, we present an essential tool in the Representation Theory as a whole.

Theorem 3.35 (Schur's lemma)**.** Let V and W be irreducible representations of a group G, and $\varphi\colon V \to W$ be a G-map between them. Then following results hold:

(1) Either φ is an isomorphism, or $\varphi = \mathbf{0}$.

(2) If $V = W$, then $\varphi = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{C}$.

Proof. Subspaces $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ of a G-map φ are G-invariant, which follows directly from Definition 3.29. Claim **(1)** then follows from irreducibility of V and W, because only possibilities are that either $\operatorname{Ker}\varphi = \{\mathbf{0}\}$ and $\operatorname{Im}\varphi = W$, or that $\operatorname{Ker}\varphi = V$ and $\operatorname{Im}\varphi = \{\mathbf{0}\}$.

Since V is a finite-dimensional vector space over algebraically closed field $\mathbb{C}$, there exists a solution $\lambda \in \mathbb{C}$ of the equation $\det(\varphi - \lambda\mathbf{1}) = 0$. Claim **(2)** then follows from **(1)** applied to a G-map $\varphi - \lambda\mathbf{1}$, giving us $\varphi - \lambda\mathbf{1} = 0 \implies \varphi = \lambda\mathbf{1}$. □

Corollary 3.36 (Dimension of Hom_G-space between irreducible representations)**.** Let V and W be two irreducible representations of a group G. Then

$$\dim \mathrm{Hom}_G(V,W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}.$$

Proof. From the claim **(1)** of Theorem 3.35 follows, that there exists a nonzero $\varphi \in \mathrm{Hom}_G(V,W)$ if and only if $V \cong W$.

In the case $V \cong W$ we fix a bijective $\varphi_0 \in \mathrm{Hom}_G(V,W)$. If also $\varphi \in \mathrm{Hom}_G(V,W)$, then $\varphi \circ \varphi_0^{-1} \in \mathrm{Hom}_G(V,V)$, and from the claim **(2)** of Theorem 3.35 follows

$$\varphi \circ \varphi_0^{-1} = \lambda\mathbf{1} \quad \text{for some } \lambda \in \mathbb{C} \quad \implies \quad \mathrm{Hom}_G(V,W) = \mathbb{C}\varphi_0. \qquad \square$$

Now we embark on the journey of proving the complete reducibility of compact Lie group representations. After that, Theorem 3.35 and mainly Corollary 3.36 will result in some far-reaching consequences.

We will exploit the existence of the invariant Haar integral to construct inner product compatible with the structure of the group. This will enable us to decompose any reducible representation to its proper subrepresentation and corresponding orthogonal complement, which will also be its subrepresentation.

Definition 3.37 (Unitary representation)**.** A representation V of a group G is called *unitary* if there exists a G-invariant sesquilinear inner product $(\bullet,\bullet)\colon V\times V\to\mathbb{C}$, that is

$$(g\boldsymbol{v},g\boldsymbol{w})=(\boldsymbol{v},\boldsymbol{w})\quad\text{for any } g\in G \text{ and } \boldsymbol{v},\boldsymbol{w}\in V.$$

Remark 3.38. Any $g\in G$ is thus represented on any unitary representation V by a *unitary* operator $g|_V$. So $g|_V$ is also a normal operator, hence *diagonalizable*.

Theorem 3.39 (Unitarity of compact Lie group representations)**.**

Let V be a representation of a compact Lie group G. Then V is unitary.

Proof. Let $\langle\bullet,\bullet\rangle$ be any inner product. We will average it over the whole G using the invariant Haar measure dg (see Theorem 3.18). Since integration preserves sesquilinearity and positive-definitness, we can define the inner product

$$(\bullet,\bullet)\equiv\int_G\langle g\bullet,g\bullet\rangle\,dg\,.$$

G-invariance of $(\bullet,\bullet)$ follows from Corollary 3.19, simply by calculation

$$(h\boldsymbol{v},h\boldsymbol{w})=\int_G\langle gh\boldsymbol{v},gh\boldsymbol{w}\rangle\,dg=\int_G\langle g\boldsymbol{v},g\boldsymbol{w}\rangle\,dg\equiv(\boldsymbol{v},\boldsymbol{w}),$$

for any $g\in G$ and $\boldsymbol{v},\boldsymbol{w}\in V$. □

Corollary 3.40 (Complete reducibility of compact Lie group representations)**.** Every representation V of a compact Lie group G is completely reducible.

Proof. Suppose V is reducible with $W\leq V$ being its proper subrepresentation. From Theorem 3.39 we obtain a G-invariant inner product $(\bullet,\bullet)$, thus we have the decomposition $V=W\oplus W^\perp$, where the orthogonal complement $W^\perp$ of W is also a proper subrepresentation of V, since

$$(g\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v},g^{-1}\boldsymbol{w})=0\quad\text{for any } g\in G,\boldsymbol{v}\in W^\perp,\boldsymbol{w}\in W.$$

We finish the proof by induction and the finite dimensionality. □

Therefore, to study any finite-dimensional representation of a compact Lie group it is enough to understand the set of irreducible representations, and to know how to compute multiplicities in its decomposition.

Corollary 3.41 (Decomposition of compact Lie group representations)**.** Let V be a representation of a compact Lie group G. Then there exists a decomposition.

$$V = \bigoplus_{i=1}^{k} V_i^{\oplus a_i} \equiv V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations with multiplicities a_i given uniquely by

$$a_i = \dim \mathrm{Hom}_G(V_i, V).$$

Proof. Existence follows from Corollary 3.41. Multiplicities are calculated as

$$\begin{aligned}\dim \mathrm{Hom}_G(V_i, V) &= \dim \mathrm{Hom}_G\left(V_i, \bigoplus_{j=1}^{k} V_j^{\oplus a_j}\right) = \dim\left[\bigoplus_{j=1}^{k} \mathrm{Hom}_G\left(V_i, V_j^{\oplus a_j}\right)\right] \\ &= \sum_{j=1}^{k} \dim \mathrm{Hom}_G\left(V_i, V_j^{\oplus a_j}\right) = \sum_{j=1}^{k} a_j \underbrace{\dim \mathrm{Hom}_G(V_i, V_j)}_{\delta_{ij} \text{ from Corollary 3.36}} = a_i,\end{aligned}$$

where we have used the isomorphism (valid for arbitrary representations U, V, W)

$$\mathrm{Hom}_G(U, V \oplus W) \cong \mathrm{Hom}_G(U, V) \oplus \mathrm{Hom}_G(U, W)$$

following from the basic linear algebra equivalence

$$U^* \otimes (V \oplus W) \cong (U^* \otimes V) \oplus (U^* \otimes W). \qquad \square$$

We will show one last consequence of Theorem 3.39, namely that dual representations of unitary representations are actually isomorphic to so-called *conjugate* representations, where multiplication by scalars from $\mathbb{C}$ is always conjugated.

Corollary 3.42 (Equivalence of $\overline{V}$ and V^*)**.** Let V be a representation of a compact Lie group G. Then we have an equivalence of representations $\overline{V} \cong V^*$, where the *conjugate* representation $\overline{V}$ has the same underlying additive structure as V, but is equipped with a new scalar multiplication structure $\cdot' \colon \mathbb{C} \times V \to V$ given by $z \cdot' \boldsymbol{v} = \overline{z}\boldsymbol{v}$.

Proof. Define a bijective linear map $\varphi : \overline{V} \to V^*$ by $\varphi \boldsymbol{v} \equiv (\boldsymbol{v}, \bullet)$ for any $\boldsymbol{v} \in V$, where $(\bullet, \bullet)$ is G-invariant inner product from Theorem 3.39. To see that φ is a G-map, and therefore realizes the equivalence $\overline{V} \cong V^*$, we calculate

$$g(\varphi \boldsymbol{v}) = (\boldsymbol{v}, g^{-1}\bullet) = (g\boldsymbol{v}, \bullet) \equiv \varphi(g\boldsymbol{v}),$$

where the first equality follows from Definition 3.27 of the action of G on V^*, and the second from G-invariance of the inner product. $\square$

Remark 3.43. From the definition of $\overline{V}$ we have $g|_{\overline{V}} \equiv \overline{g|_V}$ for any $g \in G$.

3.4 Characters and the Projection formula

We would like to explicitly compute the factors in the decomposition of a representation into the irreducible ones. The first step will be finding an explicit formula for the projection onto the direct sum of trivial factors in the decomposition. To start, we introduce a notation for this trivial factor.

Definition 3.44 (Fixed point set)**.** Let V be a representation of a group G. Then the *fixed point set* is a subrepresentation of V defined by

$$V^G \equiv \{\boldsymbol{v} \in V \mid g\boldsymbol{v} = \boldsymbol{v} \text{ for all } g \in G\}.$$

We again exploit the idea of averaging over G already used in the proof of Theorem 3.39 to obtain a G-invariant projection.

Proposition 3.45 (Projection onto V^G)**.** Let V be a representation of a compact Lie group G. Then the map $\mathsf{p} \in \mathrm{End}(V)$ defined using the vector integration for any $\boldsymbol{v} \in V$ by

$$\mathsf{p}(\boldsymbol{v}) \equiv \int_G g\boldsymbol{v}\, dg$$

is a projection of V onto V^G.

Proof. Using invariance of dg for any $h \in G$ we have

$$h\mathsf{p}(\boldsymbol{v}) \equiv \int_G hg\boldsymbol{v}\, dg = \int_G g\boldsymbol{v}\, dg \equiv \mathsf{p}(\boldsymbol{v}),$$

so $\mathrm{Im}\,\mathsf{p} \subset V^G$. Conversely, if $\boldsymbol{v} \in V^G$, then

$$\mathsf{p}(\boldsymbol{v}) \equiv \int_G g\boldsymbol{v}\, dg = \int_G \boldsymbol{v}\, dg = \boldsymbol{v}\int_G dg \equiv \boldsymbol{v},$$

so $V^G \subset \mathrm{Im}\,\mathsf{p}$ and $\mathsf{p} = \mathsf{p} \circ \mathsf{p}$ is a projection onto $\mathrm{Im}\,\mathsf{p} = V^G$. □

Remark 3.46. We can calculate the dimension of V^G using a standard property of projections, that is (we can commute Tr and integration by linearity)

$$\dim V^G = \mathrm{Tr}(\mathsf{p}) \equiv \mathrm{Tr}\left(\int_G g|_V\, dg\right) = \int_G \mathrm{Tr}(g|_V)\, dg$$

which follows from $\mathsf{p}|_{V^G} = \mathbf{1}_{V^G}$ and a convenient choice of basis adjusted to the decomposition $V = \mathrm{Ker}\,\mathsf{p} \oplus \mathrm{Im}\,\mathsf{p} = \mathrm{Ker}\,\mathsf{p} \oplus V^G$. For a visual intuition about the projection map p see Figure 3.4.

Previous remark is a straightforward demonstration, that traces of $g|_V$ appear naturally when one is exploring the structure of representations. This suggests the fundamental notion of *characters*, basis of *Character Theory*.

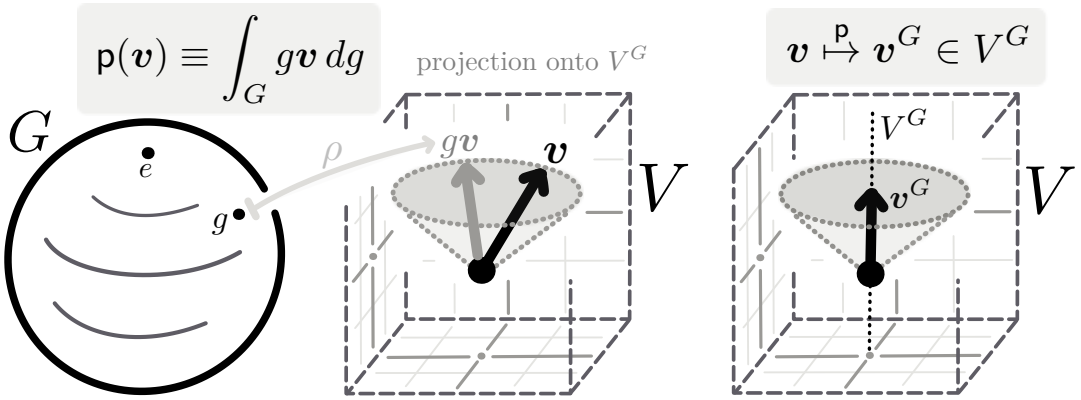


Figure 3.4: Illustration of Proposition 3.45, where we project vector $\boldsymbol{v} \in V$ onto the trivial subrepresentation V^G of the representation V of the group G. We imagine $G = \mathsf{SO}(2) \cong S^1$ and V as a 3-dimensional real representation of G, which decomposes as $V = S \oplus \mathbb{R}$, S being the standard representation on the x-y plane, and $\mathbb{R}$ being the trivial representation along the z-axis. As the projection map p averages over the whole group G, the action of $g \in G$ rotates the components of the vector in the x-y plane, thereby only leaving the component along the z-direction, which is precisely $\boldsymbol{v}^G \in V^G$.

Definition 3.47 (Character)**.** Let V be a representation of a group G. Then its *character* $\chi_V \colon G \to \mathbb{C}$ is a function on G defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V) \equiv \mathrm{Tr}_V(g) \quad \text{for any } g \in G.$$

Remark 3.48. Note that $\chi_V(e) \equiv \mathrm{Tr}(\mathbf{1}_V) = \dim V$.

Characters are condensed form of information about the representation. For finite groups and compact Lie groups they give us a unique (up to isomorphism) characterization of their finite-dimensional representations, for general Lie groups they still encode very important properties. Before we state the fundamental result concerning the calculation of multiplicities in decompositions, we first find out some useful properties of characters.

Proposition 3.49 (Properties of characters)**.** Let V and W be representations of a compact Lie group G. Then following properties hold:

(1) Characters are *class functions*, meaning they are invariant under the conjugation map, that is $\chi_V(g) = \chi_V(hgh^{-1})$ for any $g, h \in G$.

(2) If $V \cong W$, then $\chi_V = \chi_W$. Specifically, $\chi_{V^*} = \chi_{\overline{V}} = \overline{\chi_V}$.

(3) Character of direct sum $V \oplus W$ is given by $\chi_{V \oplus W} = \chi_V + \chi_W$.

(4) Character of tensor product $V \otimes W$ is given by $\chi_{V \otimes W} = \chi_V \chi_W$.

(5) Character of the dual space V^* satisfies the relation $\chi_{V^*}(g) = \chi_V(g^{-1})$.

Proof. We fix any $g, h \in G$. Claim **(1)** follows from the cyclic property of Tr.

Action on equivalent representations (see Definition 3.29) differs just by a change of basis, with respect to which the trace is invariant. Second part of claim **(2)** then follows from Corollary 3.42.

Actions of g are automatically diagonalizable on V and W (see remark 3.38). Denote by $\{\alpha_i\}$ and $\{\beta_j\}$ the eigenvalues of the action g on V and W, respectively. Using Definition 3.26 we obtain that $\{\alpha_i\} \cup \{\beta_j\}$ and $\{\alpha_i \beta_j\}$ are eigenvalues on $V \oplus W$ and $V \otimes W$, from which the claims **(3)** and **(4)** follow.

Claim **(5)** follows from Definition 3.27 and the fact that $\mathrm{Tr}(A) = \mathrm{Tr}(A^\intercal)$. □

We are now fully prepared to state the main result of this section.

Corollary 3.50 (Projection formula for the multiplicity)**.** The multiplicity a_i of the irreducible representation V_i in a decomposition of a representation V of a group G is given by

$$a_i \overset{(1)}{=} \dim \mathrm{Hom}_G(V_i, V) \overset{(2)}{=} \int_G \chi_{V_i}(g^{-1})\, \chi_V(g)\, dg \overset{(3)}{=} \int_G \overline{\chi_{V_i}(g)}\, \chi_V(g)\, dg$$

Proof. Equality **(1)** was already proved in Corollary 3.41. Equalities **(2)** and **(3)** follow from Remark 3.46 applied to the representation $\mathrm{Hom}(V_i, V)$ by noticing $\mathrm{Hom}_G(V_i, V) \equiv \mathrm{Hom}(V_i, V)^G$ (see Remark 3.30), and applying Proposition 3.49 to the equivalence $\mathrm{Hom}(V, W) \cong V^* \otimes W \cong \overline{V} \otimes W$, that is

$$\chi_{\mathrm{Hom}(V,W)}(g) = \chi_{V^* \otimes W}(g) = \chi_V(g^{-1}) \chi_W(g) = \chi_{\overline{V} \otimes W}(g) = \overline{\chi_V(g)}\, \chi_W(g). \quad \square$$

Remark 3.51. Remark 3.46 is a special case of Corollary 3.50, where we take the projection onto the trivial representation $V_i \equiv \mathbb{C}$, so $g|_{\mathbb{C}} \equiv \mathbf{1} \Longrightarrow \chi_{V_i}(g) = 1$.

Remark 3.52. *Irreducible characters* (characters of the irreducible representations) turn out to be *orthonormal*, which follows from Corollary 3.50 and Corollary 3.36.

Remark 3.53 (Quick summary)**.** In the last two sections we learned, that every finite-dimensional representation of a compact Lie group is completely reducible (Corollary 3.40). The decomposition is given uniquely (up to isomorphism) by the multiplicities of corresponding irreducible representations (Corollary 3.41), which can be calculated just using the character of the representation (Corollary 3.50). Thus, characters give us a unique characterization up to isomorphism.

3.5 Graded representations and characters

We learned a fair amount about the finite-dimensional representations, enough to be capable of extracting interesting information about the operators by assigning them appropriate representations and characters. Our goal is to find the Hilbert series $H(\boldsymbol{\phi}, \mathcal{D})$ encoding information about the operator basis containing infinite number of operators.

It is hopeless to work with one operator representation at a time, not only efficient-wise, but also because we have non-trivial IBP relations between them. Thankfully, we can organize them to so-called graded representations and calculate corresponding characters all at once.

We start by realizing that the tensor product of representations $V \otimes W$ corresponds to the representation of two distinguishable particle fields, one transforming under V, other under W. Thus, if we want n distinguishable copies of the same particle field, we just work with the nth tensor power.

Definition 3.54 (Tensor power of representation)**.** Let V be a representation of a group G. Then the nth tensor power $V^{\otimes n}$ is (by convention $V^{\otimes 0} \equiv \mathbb{C}$)

$$V^{\otimes n} \equiv \underbrace{V \otimes \cdots \otimes V}_{n} \quad \text{for } n \in \mathbb{N}.$$

But all known physical particles are in fact indistinguishable. Exchange of two particles cannot change any physical observable, so the wave-function can only change by a phase. Two subsequent exchanges of the same pair is an identity operation (ignoring peculiarities such as *anyons*), so the change of phase can only be by a factor of $\pm$, corresponding to the bosonic/fermionic particles.

Since action of the group on $V^{\otimes n}$ commutes with the permutation of the factors, fully symmetric subspace $S^n(V)$ and fully antisymmetric subspace $\bigwedge^n(V)$ are actually subrepresentations of $V^{\otimes n}$. In the following the symmetric product $\otimes_{\text{sym}}$ is just omitted and the exterior product is denoted as ordinary by the wedge $\wedge$.

Definition 3.55 (Symmetric and exterior power of representation)**.** Let V be a representation of a group G. Then the nth *symmetric power* $S^n(V)$ and the nth *exterior power* $\bigwedge^n(V)$ are subrepresentations of the nth tensor power $V^{\otimes n}$, with actions of any $g \in G$ given by

$$g(\boldsymbol{v}_1 \cdots \boldsymbol{v}_n) \equiv (g\boldsymbol{v}_1) \cdots (g\boldsymbol{v}_n), \quad g(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n) \equiv (g\boldsymbol{v}_1) \wedge \cdots \wedge (g\boldsymbol{v}_n),$$

for any $\{\boldsymbol{v}_i\}_{i=1}^n \subset V$. By convention let $S^0(V) = \bigwedge^0(V) \equiv \mathbb{C}$.

Remark 3.56. This construction is further generalized by *Schur functors* $\mathbb{S}^\lambda$ [11].

We can extend notions of tensor, symmetric, and exterior power to construct corresponding algebras. But first, we define the notion of graded representations.

Definition 3.57 (Graded representation)**.** Let V be a representation (possibly infinite-dimensional) of a group G. It is called a *graded representation* if it has the form of

$$V \equiv \bigoplus_{n=0}^{\infty} t^n V_n,$$

where V_n is a finite-dimensional representation of G for each $n \in \mathbb{N}_0$. For now, the factors t^n are just labels helping us to distinguish different graded pieces.

Remark 3.58. We defined V_n to be finite-dimensional, but we would often like them to be themselves graded representations. Standard reordering of summation

makes this requirement reasonable in the sense of Definition 3.57

$$V \equiv \bigoplus_{n=0}^{\infty} t^n V_n \equiv \bigoplus_{n=0}^{\infty} t^n \left(\bigoplus_{r=0}^{\infty} q^r W_{n,r} \right) = \bigoplus_{n=0}^{\infty} \bigoplus_{r=0}^{\infty} t^n q^r W_{n,r} = \bigoplus_{k=0}^{\infty} \overbrace{\left(\bigoplus_{n+r=k} t^n q^r W_{n,r} \right)}^{U_k},$$

where the representations $W_{n,r}$ are already finite-dimensional, thus also U_k. Alternatively, we could extend the definition of graded representations to utilize grading with multiple labels from the start.

Natural examples of graded representations arise from algebra by considering the tensor algebra and its quotients, the symmetric and the exterior algebra.

Definition 3.59 (Tensor, symmetric, and exterior graded representations)**.** Let V be a representation of a group G. We define the *tensor*, *symmetric*, and *exterior graded representations* of V, respectively, as

$$T(V) \equiv \bigoplus_{n=0}^{\infty} t^n V^{\otimes n}, \quad S(V) \equiv \bigoplus_{n=0}^{\infty} t^n S^n(V), \quad \bigwedge(V) \equiv \bigoplus_{n=0}^{\dim V} t^n \bigwedge^n(V).$$

Remark 3.60. Note that for a finite-dimensional vector space V the direct sum in the definition of the exterior graded representation terminates, because there are no non-trivial n-forms with $n > \dim V$. We still understand $\bigwedge(V)$ as a graded representation, just for $n > \dim V$ we have only trivial *zero representations*.

Remark 3.61. We say that $S(V)$ and $\bigwedge(V)$ are *graded subrepresentations* of $T(V)$, since they are subrepresentations for every graded piece.

If we want to calculate some property of a graded representation, it would be often useful to know which graded piece contributed by what amount. This is where the labels will play a fundamental role. Take for example the dimension.

Definition 3.62 (Graded dimension)**.** Let V be a graded representation of a group G. Then its *graded dimension* $\dim_t V$ is a formal series in a complex parameter t defined by

$$\dim_t V \equiv \sum_{n=0}^{\infty} t^n \dim V_n.$$

Remark 3.63. The use of labels enables us to extract useful information even in cases, when the ordinary dimension of V would be simply just infinity.

Remark 3.64. The graded dimension of V in the context of Remark 3.58 is a series with two parameters, greatly helping us to identify the former grading.

Remark 3.65 ($\mathcal{K}$ as a graded representation and $H(\phi, \mathcal{D})$ as its graded dimension)**.** The space $\mathcal{K} \equiv \operatorname{Span} \mathcal{B}$ in Definition 2.3 can be understood as a graded representation of the Lorentz (and possibly gauge or some other) group, with grading given by Definition 2.7. Every graded piece is composed of trivial representations, one

for each independent operator of a given weight, since all operators in $\mathcal{B}$ must be invariant. By comparing Definition 3.62 with Definition 2.9 we recognize that

$$\dim_{(\boldsymbol{\phi},\mathcal{D})} \mathcal{K} = H(\boldsymbol{\phi}, \mathcal{D}),$$

thus the Hilbert series H is the graded dimension of the representation $\mathcal{K} \equiv \operatorname{Span} \mathcal{B}$. That is actually why the coefficients of $H(\boldsymbol{\phi}, \mathcal{D})$ are denoted as $d_{\boldsymbol{r}n}$, because they are dimensions of the corresponding graded pieces.

For efficient manipulations it is essential to extend the definition of characters to graded representations. Graded characters will play a fundamental role in the computation of the Hilbert series.

Definition 3.66 (Graded character)**.** Let V be a graded representation of a group G. Then its *graded character* χ_V is a formal series on G in a complex parameter t defined by

$$\chi_V(t; g) \equiv \sum_{n=0}^{\infty} t^n \chi_{V_n}(g), \quad \text{for any } g \in G.$$

Remark 3.67. In particular, we have $\chi_V(t; e) = \dim_t V$, see Remark 3.48.

Remark 3.68. Similarly as in Corollary 3.50, we can use graded character χ_V to obtain *graded multiplicities* of any irreducible representation W in V as

$$\dim_t a_W \equiv \dim_t \operatorname{Hom}_G(W, V) \equiv \sum_{n=0}^{\infty} t^n \dim \operatorname{Hom}_G(W, V_n) = \int_G \overline{\chi_W(g)}\, \chi_V(t; g)\, dg\,.$$

Remark 3.69 (Alternative formulation of graded representations)**.** In the light of Definition 3.66, it appears as the nth graded piece V_n is also being scaled by the factor of t^n (in addition to the ordinary action of g on V_n represented by $g|_{V_n}$). Indeed, we can give an alternative formulation of Definition 3.57, giving a precise meaning to the labels t^n.

First, we extend our group by a *scaling group* of non-zero complex numbers with multiplication operation, that is (see Definition 3.7)

$$\mathsf{G} \equiv \mathsf{GL}(\mathbb{C}) \times G = (\mathbb{C} \setminus \{0\}) \times G \equiv \mathbb{C}^\times \times G.$$

By understanding $\mathbb{C}$ as a standard representation of $\mathsf{GL}(\mathbb{C})$ and defining representations $\mathbb{C}_n$ on $\mathbb{C}$ just by modifying the action of $t \in \mathbb{C}^\times$ to $t|_{\mathbb{C}_n} \equiv t^n$, we can distinguish different graded pieces V_n by constructing the graded representation

$$\mathsf{V} \equiv \bigoplus_{n=0}^{\infty} \mathsf{V}_n \equiv \bigoplus_{n=0}^{\infty} \mathbb{C}_n \otimes V_n,$$

with the action of $\mathsf{g} \equiv (t, g) \in \mathsf{G}$ on $\mathbf{v} \in \mathsf{V}$ defined by

$$\mathsf{g}\mathbf{v} \equiv (t; g) \left[\bigoplus_{n=0}^{\infty} \boldsymbol{v}_n \right] \equiv \bigoplus_{n=0}^{\infty} t^n g|_{V_n} \boldsymbol{v}_n, \quad \text{where } \boldsymbol{v}_n \in \mathbb{C}_n \otimes V_n.$$

Note, that V_n is isomorphic to $\mathbb{C}_n \otimes V_n$ as a complex vector space, but not as a representation of a group G. Factors of $\mathbb{C}_n$ just "charge" graded pieces V_n, granting them a scaling action. Now we can easily see, that we reproduced

$$\chi_{\mathsf{V}}(\mathsf{g}) \equiv \operatorname{Tr}(g|_{\mathsf{V}}) \equiv \operatorname{Tr}\left(\bigoplus_{n=0}^{\infty} t^n g|_{V_n}\right) = \sum_{n=0}^{\infty} t^n \chi_{V_n}(g), \quad \text{for any } \mathsf{g} \equiv (t; g) \in \mathsf{G}.$$

Since most of the time we work only with a single group element $\mathsf{g} \equiv (\boldsymbol{t}; g)$ with possibly multiple scaling parameters $\boldsymbol{t} \equiv (t_1, \dots, t_l)$, we can think of them as just being labels, just like in Definition 3.57. Nonetheless, in Corollary 3.72 we really need to use the interpretation of scaling introduced in this reformulation.

Graded characters corresponding to graded representations in Definition 3.59 can be (formally) summed up to quite pretty formulas. It is important, that we are able to find such closed forms, otherwise the calculations would be intractable.

Proposition 3.70 (Selected graded characters)**.** Let V be a finite-dimensional or graded representation of a group G. The following graded characters are given by:

$$\begin{aligned}
\chi_{T(V)}(t; g) &= \sum_{n=0}^{\infty} t^n \chi_{V^{\otimes n}}(g) = \frac{1}{1 - t \operatorname{Tr}(g|_V)} \equiv \frac{1}{1 - t\chi_V(g)} \\
\chi_{S(V)}(t; g) &= \sum_{n=0}^{\infty} t^n \chi_{S^n(V)}(g) = \frac{1}{\det(\mathbf{1} - tg|_V)} \equiv \frac{1}{\det_V(1 - tg)} \\
\chi_{\bigwedge(V)}(t; g) &= \sum_{n=0}^{\infty} t^n \chi_{\bigwedge^n(V)}(g) = \det(\mathbf{1} + tg|_V) \equiv \det_V(1 + tg)
\end{aligned}$$

Proof. We perform the proof only for a finite-dimensional V with $m \equiv \dim V$. Proof would follow along the same lines for a graded representation, only χ_V itself would be a series and V would have possibly non-finite but still countable basis.

The first relation follows from Proposition 3.49 applied to $V^{\otimes n}$, thus

$$\sum_{n=0}^{\infty} t^n \chi_{V^{\otimes n}}(g) = \sum_{n=0}^{\infty} \left[t\chi_V(g)\right]^n = \frac{1}{1 - t\chi_V(g)},$$

where the last equality is just performing the sum of a geometric series.

Fix any g and let $\{\boldsymbol{e}_i\}_{i=1}^m$ be a basis of V created from eigenvectors of $g|_V$ with corresponding eigenvalues $\{\lambda_i\}_{i=1}^m$.

For any given $n \in \mathbb{N}_0$, it is easy to see that a basis of $S^n(V)$ is

$$\left\{ \boldsymbol{e}_1^{r_1} \cdots \boldsymbol{e}_m^{r_m} \;\middle|\; r_i \in \mathbb{N}_0, \sum_{i=0}^{m} r_i = n \right\}.$$

Since we have (see Definition 3.59)

$$g(\boldsymbol{e}_1^{r_1} \cdots \boldsymbol{e}_m^{r_m}) = \lambda_1^{r_1} \cdots \lambda_m^{r_m} (\boldsymbol{e}_1^{r_1} \cdots \boldsymbol{e}_m^{r_m}),$$

it follows that

$$\sum_{n=0}^{\infty} t^n \chi_{S^n(V)}(g) = \sum_{n=0}^{\infty} t^n \sum_{|\boldsymbol{r}|=n} \boldsymbol{\lambda}^{\boldsymbol{r}} = \sum_{n=0}^{\infty} \sum_{|\boldsymbol{r}|=n} (t\boldsymbol{\lambda})^{\boldsymbol{r}} = \sum_{\boldsymbol{r}} (t\boldsymbol{\lambda})^{\boldsymbol{r}}$$
$$= \prod_{i=1}^{m} \sum_{r_i=0}^{\infty} (t\lambda_i)^{r_i} = \prod_{i=1}^{m} \frac{1}{1-t\lambda_i} = \frac{1}{\det(\mathbf{1} - tg|_V)},$$

where we used the multi-index notation $\boldsymbol{\lambda}^{\boldsymbol{r}} \equiv \prod_i \lambda_i^{r_i}$, $|\boldsymbol{r}| \equiv \sum_i r_i$.

For any given $n \in \mathbb{N}_0 \cap [0, m]$, a basis of $\bigwedge^n(V)$ is

$$\{\boldsymbol{e}_{k_1} \wedge \cdots \wedge \boldsymbol{e}_{k_n} \mid k_j \in \mathbb{N}_0, 0 \leq k_1 < \cdots < k_n \leq m\}.$$

Similarly, since we have

$$g(\boldsymbol{e}_{k_1} \wedge \cdots \wedge \boldsymbol{e}_{k_n}) = \lambda_{k_1} \cdots \lambda_{k_n} (\boldsymbol{e}_{k_1} \wedge \cdots \wedge \boldsymbol{e}_{k_n}),$$

it follows that

$$\sum_{n=0}^{m} t^n \chi_{\bigwedge^n(V)}(g) = \sum_{n=0}^{m} t^n \sum_{k_1 < \cdots < k_n} \boldsymbol{\lambda}^{\boldsymbol{k}} = \sum_{n=0}^{m} \sum_{k_1 < \cdots < k_n} (t\boldsymbol{\lambda})^{\boldsymbol{k}}$$
$$= \prod_{i=1}^{m} (1 + t\lambda_i) = \det(\mathbf{1} + tg|_V). \qquad \square$$

Remark 3.71. Summed up graded characters are reminiscent of *partition functions* with corresponding statistics found in statistical mechanics, namely the grand-canonical partition functions for Bose-Einstein/Fermi-Dirac ideal quantum gases

$$\mathcal{Z}_g = \prod_n \left[1 \mp e^{-\beta(E_n - \mu)}\right]^{\mp} \equiv \prod_n \left[1 \mp z e^{-\beta E_n}\right]^{\mp},$$

where n goes through all energy states with energies E_n, and as usual $\beta \equiv \frac{1}{k_B T}$. The *fugacity* $z \equiv e^{\beta\mu}$ plays the role of t, and $e^{-\beta E_n}$ plays the role of λ_i.

There is another way to express graded characters of $S(V)$ and $\bigwedge(V)$, mainly useful in the case when V itself is a graded representation. It enables us to sidestep the calculation of determinant just by knowledge of its graded character.

Corollary 3.72 (Graded characters as a Plethystic Exponential)**.** Let V be a graded representation of a group G with a label q. Graded characters in Proposition 3.70 can be also calculated as

$$\chi_{S(V)}(t, q; g) = \exp\left(\sum_{r=1}^{\infty} (+1)^{r+1} \frac{t^r}{r} \chi_V(q^r; g^r)\right) \equiv \mathrm{PE}\left[t\chi_V(q; g)\right],$$
$$\chi_{\bigwedge(V)}(t, q; g) = \exp\left(\sum_{r=1}^{\infty} (-1)^{r+1} \frac{t^r}{r} \chi_V(q^r; g^r)\right) \equiv \mathrm{PE}_f\left[t\chi_V(q; g)\right].$$

Proof. First, we perform manipulations with an arbitrary matrix of the form $\mathbf{1} \mp tA$, where we a use well-known formula $\det \exp = \exp \operatorname{Tr} \Rightarrow \det = \exp \operatorname{Tr} \ln$, and expansion of ln around $\mathbf{1}$, to obtain

$$\begin{aligned}[\det(\mathbf{1} \mp tA)]^{\mp} &= [\exp \operatorname{Tr} \ln(\mathbf{1} \mp tA)]^{\mp} \\ &= \exp\left[\mp \operatorname{Tr}\left(\sum_{r=1}^{\infty}(-1)^{r+1}\frac{(\mp tA)^r}{r}\right)\right] = \exp\left(\sum_{r=1}^{\infty}(\pm 1)^{r+1}\frac{t^r}{r}\operatorname{Tr}(A^r)\right).\end{aligned}$$

Since the action of g on V_n is given by $q^n g|_{V_n}$ (see Remark 3.69), the result follows from Proposition 3.70 and calculation

$$\begin{aligned}&[\det_{\mathsf{V}}(1 \mp tg)]^{\mp} = \left[\prod_{n=0}^{\infty}\det_{\mathsf{V}_n}(1 \mp tg)\right]^{\mp} \equiv \prod_{n=0}^{\infty}[\det(\mathbf{1} \mp tq^n g|_{V_n})]^{\mp} = \text{[from before]} \\ &= \prod_{n=0}^{\infty}\exp\left(\sum_{r=1}^{\infty}(\pm 1)^{r+1}\frac{t^r}{r}\operatorname{Tr}[(q^n g|_{V_n})^r]\right) = \exp\left(\sum_{n=0}^{\infty}\sum_{r=1}^{\infty}(\pm 1)^{r+1}\frac{t^r}{r}\operatorname{Tr}[q^{nr}g^r|_{V_n}]\right) \\ &= \exp\left(\sum_{r=1}^{\infty}(\pm 1)^{r+1}\frac{t^r}{r}\sum_{n=0}^{\infty}(q^r)^n\operatorname{Tr}[g^r|_{V_n}]\right) \equiv \exp\left(\sum_{r=1}^{\infty}(\pm 1)^{r+1}\frac{t^r}{r}\chi_V(q^r;g^r)\right),\end{aligned}$$

or more simply using $\mathsf{g}^r = (q;g)^r = (q^r;g^r)$ as

$$[\det_{\mathsf{V}}(1 \mp tg)]^{\mp} = \exp\left(\sum_{r=1}^{\infty}(\pm 1)^{r+1}\frac{t^r}{r}\chi_{\mathsf{V}}(\mathsf{g}^r)\right) \equiv \exp\left(\sum_{r=1}^{\infty}(\pm 1)^{r+1}\frac{t^r}{r}\chi_V(q^r;g^r)\right). \quad \square$$

Remark 3.73 (Plethystic Exponential)**.** For a function $\alpha(t_1,\dots,t_k)$ satisfying $\alpha(0,\dots,0) = 0$, the (fermionic) Plethystic Exponential is defined by [12]

$$\mathrm{PE}_f[\alpha(t_1,\dots,t_k)] \equiv \exp\left(\sum_{r=1}^{\infty}(-1)^{r+1}\frac{1}{r}\alpha(t_1^r,\dots,t_k^r)\right).$$

It is customary to leave out f in PE_f, and just automatically include the factor $(-1)^{r+1}$ for "fermionic" functions. Already working with this convention, for two functions α, β (either one bosonic or fermionic) of some common variables, the Plethystic Exponential satisfies the *sum-to-product* property

$$\mathrm{PE}[\alpha + \beta] = \mathrm{PE}[\alpha]\,\mathrm{PE}[\beta].$$

One simple example is $\alpha(t,q) \equiv t$ and $\beta(t,q) \equiv q$, both bosonic or both fermionic, for which it generates all anti-symmetric combinations, that is

$$\mathrm{PE}[t+q] = \mathrm{PE}[t]\,\mathrm{PE}[q] = \begin{cases} \frac{1}{(1-t)(1-q)} = 1 + t + q + t^2 + tq + q^2 + \cdots, \\ (1+t)(1+q) = 1 + t + q + tq, \end{cases}$$

thus reflecting the corresponding statistics.

3.6 Weyl integration formula

A particularly hard question is, how do we actually compute integrals appearing in Corollary 3.50. Normally, we would need to have a parametrization of a

compact Lie group G, construct the invariant Haar measure dg by (for example) left-translating the volume form in the identity of G to the whole G, express characters in a given parametrization, and perform the D-dimensional integral, where D is dimension of G. More often than not, immediately the first part of finding the parametrization can be extremely tough.

But there exists a remedy, which by utilizing the fact that characters are *class functions* and a deep analysis of the structure of compact Lie groups enables us to restrict the integration to a simple subgroup of G, moreover preparing us just right to use the powerful method of complex integration.

But first, we must expand our toolbox. Mostly we will just quickly recall some elementary definitions, since one can find more details in any book on Lie groups. Already in Proposition 3.16 we found the isomorphism between left-invariant vector fields and vectors at the identity of a Lie group. Furthermore, this enables us to induce algebra structure on the tangent space at identity.

Definition 3.74 (Lie algebra)**.** A *Lie algebra* $\mathfrak{g} \equiv \boldsymbol{T}_e G$ of a Lie group G is the tangent vector space at the identity of G, together with the *algebra structure*

$$[\boldsymbol{a}, \boldsymbol{b}] \equiv [\boldsymbol{\ell}_{\boldsymbol{a}}, \boldsymbol{\ell}_{\boldsymbol{b}}]|_e \quad \text{for any } \boldsymbol{a}, \boldsymbol{b} \in \mathfrak{g},$$

induced from the *Lie bracket* of left invariant vector fields on G.

Remark 3.75 (Jacobi identity)**.** We automatically obtain the *Jacobi identity*

$$[\boldsymbol{a}, [\boldsymbol{b}, \boldsymbol{c}]] + [\boldsymbol{c}, [\boldsymbol{a}, \boldsymbol{b}]] + [\boldsymbol{b}, [\boldsymbol{c}, \boldsymbol{a}]] = \boldsymbol{0}, \quad \text{for any } \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathfrak{g},$$

since it holds for the Lie bracket of vector fields.

Remark 3.76. Consider the behaviour of group multiplication $m\colon G \times G \to G$ near identity. The corresponding pushforward at the point $(e; e)$ is given by

$$m_*\Big|_{(e;e)}\colon\ \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}, \quad \boldsymbol{a} \oplus \boldsymbol{b} \mapsto \boldsymbol{a} + \boldsymbol{b}.$$

This can be seen by considering curve $\gamma^{\boldsymbol{a}}_\bullet\colon \mathbb{R} \to G$ with the tangent vector $\boldsymbol{a}$ at the identity $e \equiv \gamma_{\tau=0}$. We denote this by $\boldsymbol{a} \equiv \frac{\boldsymbol{D}}{d\tau}\gamma_\tau\Big|_{\tau=0}$. Then, specially choosing combination $\boldsymbol{a} \oplus \boldsymbol{0} \leftrightarrow (\gamma^{\boldsymbol{a}}_\bullet; e)$ we have

$$m_*\Big|_{(e;e)}(\boldsymbol{a} \oplus \boldsymbol{0}) \equiv m_*\Big|_{(e;e)} \frac{\boldsymbol{D}}{d\tau}(\gamma^{\boldsymbol{a}}_\tau; e)\Big|_{\tau=0} = \frac{\boldsymbol{D}}{d\tau} m(\gamma^{\boldsymbol{a}}_\tau; e)\Big|_{\tau=0} = \frac{\boldsymbol{D}}{d\tau}\gamma^{\boldsymbol{a}}_\tau\Big|_{\tau=0} \equiv \boldsymbol{a}.$$

The result follows from an analogous argument for $\boldsymbol{b}$, linearity of pushforward (general property of pushforwards), and finally taking $\boldsymbol{a} \oplus \boldsymbol{b} = \boldsymbol{a} \oplus \boldsymbol{0} + \boldsymbol{0} \oplus \boldsymbol{b}$.

Intuitively, near identity the group multiplication of G is reflected in the linear structure of vector addition in $\mathfrak{g}$. This is very useful for the study of Lie groups.

It is easy to go from G to $\mathfrak{g}$, we just have to look at the tangent vectors. After understanding the "small" transformations of G near identity, it is also desirable to go in the opposite way. There indeed exists the natural *exponential map*.

Definition 3.77 (Exponential map)**.** Let $\mathfrak{g}$ be a Lie algebra of a Lie group G. The *exponential map* is defined by

$$e^{\bullet} \equiv \exp(\bullet)\colon\ \mathfrak{g} \to G, \quad \boldsymbol{a} \mapsto \gamma_1^{\boldsymbol{a}} \equiv \gamma^{\boldsymbol{a}}(1),$$

where $\gamma_{\bullet}^{\boldsymbol{a}} : \mathbb{R} \to G$ is the integral curve of the left-invariant vector field $\boldsymbol{\ell}_{\boldsymbol{a}}$ (corresponding to $\boldsymbol{a}$ by Proposition 3.16) going through the identity $e \equiv \gamma^{\boldsymbol{a}}_{\tau=0}$.

Since curves $\sigma \mapsto \gamma^{\tau\boldsymbol{a}}(\sigma)$ and $\sigma \mapsto \gamma^{\boldsymbol{a}}(\tau\sigma)$ actually correspond to the same vector $\tau\boldsymbol{a} \in \mathfrak{g}$, they are equal, and we obtain

$$e^{\tau\boldsymbol{a}} \equiv \gamma^{\tau\boldsymbol{a}}(1) = \gamma^{\boldsymbol{a}}(\tau) \equiv \gamma^{\boldsymbol{a}}_{\tau} \quad \Longrightarrow \quad \boldsymbol{a} \equiv \frac{\boldsymbol{D}}{d\tau} e^{\tau\boldsymbol{a}}\Big|_{\tau=0}.$$

From the properties of integral curves we also obtain following

$$e^{(\tau+\sigma)\boldsymbol{a}} = \gamma^{\boldsymbol{a}}_{\tau+\sigma} = \gamma^{\boldsymbol{a}}_{\tau}\gamma^{\boldsymbol{a}}_{\sigma} = e^{\tau\boldsymbol{a}}e^{\sigma\boldsymbol{a}},\ e^{0\boldsymbol{a}} = e^{\boldsymbol{0}} \equiv e \quad \Longrightarrow \quad (e^{\tau\boldsymbol{a}})^{-1} = e^{-\tau\boldsymbol{a}}.$$

Remark 3.78. It can be easily seen, that the differential (or pushforward) of the exponential map at the origin $\boldsymbol{0} \in \mathfrak{g}$ is the identity, thus by the *inverse function theorem* it is locally invertible at $\boldsymbol{0} \mapsto e$. For an illustration see Figure 3.5.

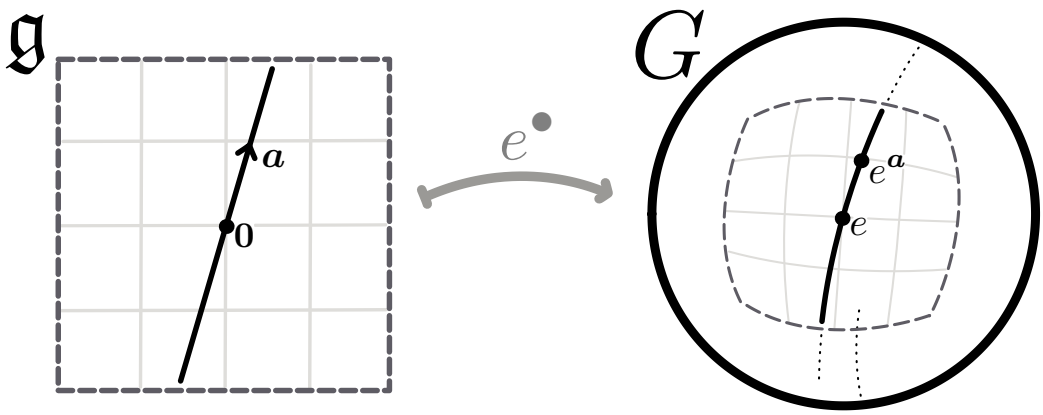


Figure 3.5: Illustration of the exponential map $e^{\bullet}$ mapping from the Lie algebra $\mathfrak{g}$ to the corresponding Lie group G. Locally around $\boldsymbol{0} \in \mathfrak{g}$ it is a diffeomorphism.

Similarly as groups, we can also represent Lie algebras on other vector spaces.

Definition 3.79 (Lie algebra representation)**.** A representation of a Lie algebra $\mathfrak{g}$ on a finite-dimensional vector space V is a homomorphism of Lie algebras $\xi_{\bullet}\colon\ \mathfrak{g} \to \mathrm{End}(V)$, meaning it preserves the Lie algebra structure, that is

$$\xi_{[\boldsymbol{a},\boldsymbol{b}]} = [\xi_{\boldsymbol{a}}, \xi_{\boldsymbol{b}}] \equiv \xi_{\boldsymbol{a}} \cdot \xi_{\boldsymbol{b}} - \xi_{\boldsymbol{b}} \cdot \xi_{\boldsymbol{a}} \quad \text{for any } \boldsymbol{a}, \boldsymbol{b} \in \mathfrak{g},$$

where the right side can be understood as the matrix commutator.

Remark 3.80. Note, that $\mathrm{End}(V)$ is a Lie algebra of $\mathsf{GL}(V)$.

In the following definition we introduce the notion of *inner automorphism* or *conjugation*, and the related *Adjoint* and *adjoint* representations. A great deal of information about a Lie group can be extracted by study of their structure.

Definition 3.81 (Conjugation, Adjoint and adjoint representations)**.** Let G be a Lie group. For any element $g \in G$ we have a *conjugation* automorphism $\mathrm{AD}_g \colon G \to G$ given by $\mathrm{AD}_g \equiv L_g R_{g^{-1}}$, that is for any $h \in G$ we have

$$\mathrm{AD}_g h \equiv ghg^{-1}, \text{ also } \mathrm{AD}_g \mathrm{AD}_h = \mathrm{AD}_{gh}.$$

In the following let $\gamma_\bullet \colon \mathbb{R} \to G$ be a curve with the tangent vector $\boldsymbol{a}$ at the identity $e \equiv \gamma_{\tau=0}$. For example we can take $\gamma_\tau \equiv e^{\tau \boldsymbol{a}}$.

Corresponding induced map on the vectors at the identity gives rise to the *Adjoint* representation $\mathrm{Ad}_\bullet \colon G \to \mathsf{GL}(\mathfrak{g})$ of G on $\mathfrak{g}$, that is $\mathrm{Ad}_g \equiv \mathrm{AD}_{g*}|_e$. Equivalently, using the previously defined curve we have

$$\mathrm{Ad}_g \boldsymbol{a} \equiv \frac{\boldsymbol{D}}{d\tau} \mathrm{AD}_g \gamma_\tau \Big|_{\tau=0}.$$

Once more, the Adjoint representation induces the *adjoint* representation $\mathrm{ad}_\bullet \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ of $\mathfrak{g}$ on $\mathfrak{g}$ by $\mathrm{ad}_\bullet \equiv \mathrm{Ad}_{\bullet *}|_e$, that is for any $\boldsymbol{b} \in \mathfrak{g}$ we have

$$\mathrm{ad}_{\boldsymbol{a}} \boldsymbol{b} \equiv \frac{d}{d\tau} \mathrm{Ad}_{\gamma_\tau} \boldsymbol{b} \Big|_{\tau=0},$$

where we write "normal" derivative, because if V is a finite-dimensional vector space, then $\boldsymbol{T}_p V$ is canonically isomorphic to V for any $p \in V$.

Remark 3.82 (Adjoint representation $\mathrm{Ad}_\bullet$ is orthogonal)**.** From construction $\mathfrak{g}$ is a real vector space. Hence, the Adjoint map $\mathrm{Ad}_\bullet$ is a *real* representation. Indeed, induced maps are in general linear, and Ad_g is invertible with inverse $\mathrm{Ad}_{g^{-1}}$, since

$$\mathrm{Ad}_g \mathrm{Ad}_{g^{-1}} = \mathrm{AD}_{g*} \mathrm{AD}_{g^{-1}*} \Big|_e = (\mathrm{AD}_g \mathrm{AD}_{g^{-1}})_* \Big|_e = (\mathrm{AD}_{gg^{-1}})_* \Big|_e = \mathrm{AD}_{e*} \Big|_e = \mathbf{1},$$

and also satisfies property $\mathrm{Ad}_g \mathrm{Ad}_h = \mathrm{Ad}_{gh}$, because $\mathrm{AD}_g \mathrm{AD}_h = \mathrm{AD}_{gh}$.

Additionally, we can construct a real scalar product on $\mathfrak{g}$ that is Ad-invariant, by similar arguments as in Theorem 3.39. We just take any inner product $\langle \bullet, \bullet \rangle$ on $\mathfrak{g}$ and average it over G using invariant Haar measure dg as

$$(\bullet, \bullet) \equiv \int_G \langle \mathrm{Ad}_g \bullet, \mathrm{Ad}_g \bullet \rangle \, dg \quad \Longrightarrow \quad (\mathrm{Ad}_h \bullet, \mathrm{Ad}_h \bullet) = (\bullet, \bullet) \quad \text{for any } h \in G.$$

Thus, the Adjoint representation $\mathrm{Ad}_\bullet$ is *orthogonal* (see Remark 3.12).

Remark 3.83. One can show that the action of adjoint representation is actually given by the Lie bracket, that is $\mathrm{ad}_{\boldsymbol{a}} \boldsymbol{b} = [\boldsymbol{a}, \boldsymbol{b}]$. Hence, we can rewrite the Jacobi identity in Remark 3.75 as $\mathrm{ad}_{[\boldsymbol{a},\boldsymbol{b}]} = [\mathrm{ad}_{\boldsymbol{a}}, \mathrm{ad}_{\boldsymbol{b}}] \equiv \mathrm{ad}_{\boldsymbol{a}} \cdot \mathrm{ad}_{\boldsymbol{b}} - \mathrm{ad}_{\boldsymbol{b}} \cdot \mathrm{ad}_{\boldsymbol{a}}$, which confirms that $\mathrm{ad}_\bullet$ is really a representation in the sense of Definition 3.79.

We also have very useful formulas relating the exponential and adjoint maps. They can be recognized as common identities valid for matrices in linear algebra, but they also hold generally in the abstract setting of Lie groups and algebras.

Proposition 3.84 (Formulas for $e^\bullet$, $\mathrm{AD}_\bullet$, $\mathrm{Ad}_\bullet$, and $\mathrm{ad}_\bullet$)**.** Let $\mathfrak{g}$ be a Lie algebra of a Lie group G. Then for any $\boldsymbol{a} \in \mathfrak{g}$ and $g \in G$ we have

$$\mathrm{AD}_g e^{\boldsymbol{a}} = e^{\mathrm{Ad}_g \boldsymbol{a}} \quad \text{and} \quad \mathrm{Ad}_{e^{\boldsymbol{a}}} = e^{\mathrm{ad}_{\boldsymbol{a}}},$$

where $e^{\mathrm{ad}_{\boldsymbol{a}}}$ is understood as a matrix exponential of $\mathrm{ad}_{\boldsymbol{a}} \in \mathrm{End}(\mathfrak{g})$.

Proof. It is enough to check the following properties of exponential

$$\mathrm{AD}_g e^{(\tau+\sigma)\boldsymbol{a}} = \mathrm{AD}_g(e^{\tau\boldsymbol{a}} e^{\sigma\boldsymbol{a}}) = g e^{\tau\boldsymbol{a}} g^{-1} g e^{\sigma\boldsymbol{a}} g^{-1} = (\mathrm{AD}_g e^{\tau\boldsymbol{a}})(\mathrm{AD}_g e^{\sigma\boldsymbol{a}}),$$

$$\mathrm{AD}_g e^{0\boldsymbol{a}} = \mathrm{AD}_g e^{\boldsymbol{0}} = \mathrm{AD}_g e = e, \quad \frac{\boldsymbol{D}}{d\tau} \mathrm{AD}_g e^{\tau\boldsymbol{a}}\Big|_{\tau=0} = \mathrm{Ad}_g \frac{\boldsymbol{D}}{d\tau} e^{\tau\boldsymbol{a}}\Big|_{\tau=0} = \mathrm{Ad}_g \boldsymbol{a}.$$

Similarly, for the second identity we calculate

$$\mathrm{Ad}_{e^{(\tau+\sigma)\boldsymbol{a}}} = \mathrm{Ad}_{e^{\tau\boldsymbol{a}} e^{\sigma\boldsymbol{a}}} = \mathrm{Ad}_{e^{\tau\boldsymbol{a}}} \mathrm{Ad}_{e^{\sigma\boldsymbol{a}}},$$

$$\mathrm{Ad}_{e^{0\boldsymbol{a}}} = \mathrm{Ad}_{e^{\boldsymbol{0}}} = \mathrm{Ad}_e = \mathbf{1}, \quad \frac{\boldsymbol{D}}{d\tau} \mathrm{Ad}_{e^{\tau\boldsymbol{a}}}\Big|_{\tau=0} = \mathrm{ad}_{\frac{\boldsymbol{D}}{d\tau} e^{\tau\boldsymbol{a}}\Big|_{\tau=0}} = \mathrm{ad}_{\boldsymbol{a}}. \qquad \square$$

In Example 3.8 we presented a simple example of a Lie group, namely a torus $\mathbb{T}^n$. Its simplicity stems from the commutative multiplication of complex numbers. Since any neighborhood of identity already generates a compact connected Lie group, this implies that exponential map is surjective on torus [8].

Perhaps surprisingly, we will be always able to restrict the integration of class functions on a compact connected Lie group to its *maximal torus*.

Definition 3.85 (Maximal torus, rank, Cartan subalgebra)**.** Let G be a compact connected Lie group with the Lie algebra $\mathfrak{g}$. Then we define:

(1) A subgroup $T \subset G$ is a *maximal torus* of G if it is a torus in the sense of Example 3.8, and there is no other torus T' with $T \subsetneq T' \subset G$. Equivalently, T is a maximal connected abelian subgroup of G, that is

$$gh = hg \quad \text{for any } g, h \in T.$$

The dimension of the maximal torus T in G is called the *rank* of G.

(2) A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is a *Cartan subalgebra* of $\mathfrak{g}$ if it is a Lie algebra of some maximal torus T of G, see **(1)**. Equivalently, $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$, that is

$$[\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{0}, \quad \text{for any } \boldsymbol{a}, \boldsymbol{b} \in \mathfrak{t}.$$

Remark 3.86. Various alternative definitions must be proved to be equivalent. An interested reader can find the proofs for example in [8, 10]. They fundamentally exploit properties of the exponential map (see Definition 3.77 and Proposition 3.84).

Example 3.87 (Maximal torus of $\mathsf{SO(d)}$)**.** The maximal torus of $\mathsf{SO(d)}$ is

$$T_{\mathsf{SO(d)}} = \underbrace{\mathsf{SO}(2) \times \cdots \times \mathsf{SO}(2)}_{\mathsf{r}} \cong (S^1)^{\mathsf{r}}.$$

where $\mathsf{r} \equiv \lfloor \mathsf{d}/2 \rfloor$ is the rank of $\mathsf{SO(d)} \equiv \mathsf{SO}(2\mathsf{r}+1)$. The torus $T_{\mathsf{SO(d)}}$ can be parametrized in the standard representation $\square \equiv \mathbb{C}^{\mathsf{d}} \equiv \mathbb{C}^{2\mathsf{r}+1}$ as

$$T = \left\{ \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & & \\ \sin\theta_1 & \cos\theta_1 & & & & \\ & & \ddots & & & \\ & & & \cos\theta_{\mathsf{r}} & -\sin\theta_{\mathsf{r}} & \\ & & & \sin\theta_{\mathsf{r}} & \cos\theta_{\mathsf{r}} & \\ & & & & & 1 \end{pmatrix} \middle| \begin{array}{l} \\ \\ \theta_j \in [0, 2\pi) \\ j \in 1, \ldots, \mathsf{r} \\ \\ \\ \end{array} \right\}$$

Alternatively, we can parametrize by r complex variables on the unit circle, namely by $\boldsymbol{x} \equiv (x_1, \ldots, x_{\mathsf{r}}) \equiv (e^{i\theta_1}, \ldots, e^{i\theta_{\mathsf{r}}})$.

Definition 3.88 (Weyl group)**.** Let T be a maximal torus of a compact connected Lie group G, and N be the normalizer of T in G, that is

$$N = \left\{ g \in G \;\middle|\; gTg^{-1} = T \right\}.$$

Then the quotient group $\mathfrak{W} \equiv N/T$ is called the *Weyl group* of G.

Remark 3.89. It can be shown that the Weyl group $\mathfrak{W}$ is always finite. Intuitively, it is a group that permutes the factors in the torus.

We are now ready to state one of the main theorems of this section. It is generalization of a well-known fact, that any $\mathsf{SO(d)}$ matrix can be conjugated by a change of basis to the corresponding torus.

Theorem 3.90 (Maximal torus theorem)**.** Let T be a maximal torus of a connected compact Lie group G. Then every element $g \in G$ is *conjugate* to an element of T, that is

$$\exists\, h \in G, t \in T\colon \;\; g = \mathrm{AD}_h t \equiv hth^{-1}.$$

Moreover, a general element of G is conjugate to $|\mathfrak{W}|$ such elements of T.

Remark 3.91. One of the approaches (see [8]) to proving this important theorem studies the map Ψ defined in the "proof" of Theorem 3.97.

Example 3.92 (Character of the standard reprentation $\square$ of $\mathsf{SO(d)}$)**.** Since characters are *class functions*, and by Theorem 3.90 every element can be conjugated to the maximal torus, to evaluate $\chi(g)$ it is enough to specify any corresponding torus element $t \leftrightarrow g$. Using parametrization introduced in Example 3.87 we obtain

$$\chi_{\square}(\boldsymbol{x} \leftrightarrow g) = \begin{cases} 2\sum_{i=1}^{\mathsf{r}} \cos(\theta_i) = \sum_{i=1}^{\mathsf{r}} \left(x_i + \frac{1}{x_i} \right) & \text{for } \mathsf{d} = 2\mathsf{r}, \\ 1 + 2\sum_{i=1}^{\mathsf{r}} \cos(\theta_i) = 1 + \sum_{i=1}^{\mathsf{r}} \left(x_i + \frac{1}{x_i} \right) & \text{for } \mathsf{d} = 2\mathsf{r}+1, \end{cases}$$

where $\square \equiv \mathbb{C}^{\mathsf{d}}$ is the standard representation of $\mathsf{SO}(\mathsf{d})$.

Example 3.93 (Graded character of $S(\square)$ for $\mathsf{SO}(\mathsf{d})$)**.** Similarly as in previous example, using Proposition 3.70, Example 3.87, and realizing that eigenvalues of the torus element $t \leftrightarrow \boldsymbol{x}$ are exactly given by $\{1, x_i, 1/x_i\}_{i=1}^{\mathsf{r}}$, we obtain

$$P(t; \boldsymbol{x} \leftrightarrow g) \equiv \chi_{S(\square)}(t; \boldsymbol{x} \leftrightarrow g) = \begin{cases} \prod_{i=1}^{\mathsf{r}} \frac{1}{(1-tx_i)(1-t/x_i)} & \text{for } \mathsf{d} = 2\mathsf{r}, \\ \frac{1}{1-t} \prod_{i=1}^{\mathsf{r}} \frac{1}{(1-tx_i)(1-t/x_i)} & \text{for } \mathsf{d} = 2\mathsf{r}+1. \end{cases}$$

We denoted $P \equiv \chi_{S(\square)}$, since it will later turn out to be the *"projection"* factor.

Maximal torus theorem shows the significance of the conjugation map $\mathrm{AD}_\bullet$. Naturally, it is easier to study its "linearized" version, the Adjoint representation $\mathrm{Ad}_\bullet$. If we consider only operators on $\mathfrak{g}$ corresponding to the torus elements, that is $\{\mathrm{Ad}_t \mid t \in T\}$, and recalling that $\mathrm{Ad}_\bullet$ is orthogonal (see Remark 3.82), we obtain a commuting set of normal operators. Therefore, it will be possible to perform *simultaneous diagonalization*, but first we must *complexify* $\mathfrak{g}$.

Definition 3.94 (Complexification of Lie algebra)**.** Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. The *complexification* of $\mathfrak{g}$ is defined by $\mathfrak{g}_{\mathbb{C}} \equiv \mathfrak{g} \otimes \mathbb{C}$, where the algebra structure $[\bullet, \bullet]$ of $\mathfrak{g}$ is extended on $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity. Similarly, we extend representations $(\mathrm{Ad}, \mathfrak{g})$ and $(\mathrm{ad}, \mathfrak{g})$ to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity.

Definition 3.95 (Roots of Lie algebra)**.** Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G of rank r, T be a maximal torus of G, and $\mathfrak{t}$ be the corresponding Cartan subalgebra of $\mathfrak{g}$. There is a set of nonzero elements $\mathrm{R}(G) \subset \mathfrak{t}_{\mathbb{C}}^*$, called the *roots* of $\mathfrak{g}_{\mathbb{C}}$, such that we obtain decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathrm{R}(G)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is nonzero space called the *root space* corresponding to α defined by

$$\begin{aligned} \mathfrak{g}_\alpha &\equiv \{\boldsymbol{a} \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}_{\boldsymbol{m}} \boldsymbol{a} = [\boldsymbol{m}, \boldsymbol{a}] = \alpha(\boldsymbol{m})\boldsymbol{a} \ \text{ for } \boldsymbol{m} \in \mathfrak{t}\} \\ &= \left\{ \boldsymbol{a} \in \mathfrak{g}_{\mathbb{C}} \;\middle|\; \mathrm{Ad}_t \boldsymbol{a} = \mathrm{Ad}_{e^{\boldsymbol{m}}} \boldsymbol{a} = e^{\alpha(\boldsymbol{m})} \boldsymbol{a} \ \text{ for } t \equiv e^{\boldsymbol{m}} \in T \right\}, \end{aligned}$$

where we used Proposition 3.84. Choosing an orthonormal basis for $\mathfrak{t}$ and representing $\alpha \leftrightarrow \boldsymbol{\alpha} \equiv (\alpha_1, \dots, \alpha_{\mathsf{r}})$ and $t \leftrightarrow \boldsymbol{x} \equiv (x_1, \dots, x_{\mathsf{r}}) \equiv (e^{m_1}, \dots, e^{m_{\mathsf{r}}})$ in the spirit of Example 3.87, we also obtain

$$e^{\alpha(\boldsymbol{m})} = e^{\sum_{i=1}^{\mathsf{r}} \alpha_i m_i} = \prod_{i=1}^{\mathsf{r}} (e^{m_i})^{\alpha_i} \equiv \prod_{i=1}^{\mathsf{r}} x_i^{\alpha_i} \equiv \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Remark 3.96. We will use the standard fact that $R(G)$ is symmetric about the origin, that is if $\boldsymbol{\alpha} \in R(G)$ is a root, then $-\boldsymbol{\alpha} \in R(G)$ is a root as well. Since roots are nonzero, there is always an even number of roots.

Finally, we are prepared to state and partially prove the main theorem of this section — the *Weyl integration formula*. Together with the *maximal torus theorem* they open the gates to the deeper representation theory of compact Lie groups.

Theorem 3.97 (Weyl integration formula)**.** Let f be a class function on a connected compact Lie group G of rank r, T be a maximal torus of G parametrized by $\boldsymbol{x} \equiv (x_1, \dots, x_{\mathsf{r}})$, and $\mathfrak{W}$ be the corresponding Weyl group. Then we have

$$\int_G f(g)\, dg = \frac{1}{|\mathfrak{W}|} \oiint\limits_{|x_i|=1} f(\boldsymbol{x}) \left[\prod_{\boldsymbol{\alpha} \in \mathrm{R}(G)} (1 - \boldsymbol{x}^{\boldsymbol{\alpha}}) \right] \left[\prod_{i=1}^{\mathsf{r}} \frac{dx_i}{2\pi i x_i} \right].$$

Remark 3.98. We use the same notation f for a class function on G, for its restriction to T, and also for its coordinate expression.

Sketch of the proof. The idea is to restrict the integral of f over the whole G only to the suitable integral over T. Consider the map

$$\Psi \colon G/T \times T \to G, \quad (gT; t) \mapsto \Psi_{gT}(y) \equiv \Psi(gT; t) \equiv gtg^{-1}.$$

From Theorem 3.90 we know, that Ψ is a finite-sheeted covering of G with $|\mathfrak{W}|$ sheets, where $\mathfrak{W}$ is the Weyl group of G. Hence, it follows that

$$\int_G f\, dg \equiv \int_G f \boldsymbol{\omega}_G = \int_{\Psi(G/T \times T)} f \boldsymbol{\omega}_G = \frac{1}{|\mathfrak{W}|} \int_{G/T \times T} \Psi^*(f \boldsymbol{\omega}_G).$$

Since f is a class function, from definition of Ψ we have

$$(\Psi^* f)|_{(gT;t)} = f \circ \Psi(gT, t) = f(gtg^{-1}) = f(t),$$

thus only thing left to calculate is $\Psi^* \boldsymbol{\omega}_G$.

To avoid dealing with many technicalities, we assume there exist left invariant normalized volume forms $\boldsymbol{\omega}_{G/T}$ on G/T and $\boldsymbol{\omega}_T$ on T. In the following denote by π_1 and π_2 the natural projection maps from $G/T \times T$ to G/T and T, respectively. We obtain a left-invariant volume form on $G/T \times T$ as

$$\boldsymbol{\omega}_{G/T \times T} \equiv (\pi_1^* \boldsymbol{\omega}_{G/T}) \wedge (\pi_2^* \boldsymbol{\omega}_T).$$

We want to find the factor $\mathfrak{D}$ relating the volume forms $\Psi^* \boldsymbol{\omega}_G$ and $\boldsymbol{\omega}_{G/T \times T}$, that is for any $(gT; t) \in G/T \times T$

$$(\Psi^* \boldsymbol{\omega}_G)\Big|_{(gT;t)} = \mathfrak{D}(gT; t) \Big[(\pi_1^* \boldsymbol{\omega}_{G/T}) \wedge (\pi_2^* \boldsymbol{\omega}_T) \Big]\Big|_{(gT;t)}.$$

Using the left-invariance of $\boldsymbol{\omega}_\bullet$ we can translate the calculation to the identity by

$$\begin{aligned} (\Psi^* \boldsymbol{\omega}_G)\Big|_{(gT;t)} &\equiv \Psi^*(\boldsymbol{\omega}_G|_{gtg^{-1}}) = \Psi^* L^*_{gt^{-1}g^{-1}} \boldsymbol{\omega}_G|_e \\ &= L^*_{(g^{-1};t^{-1})} \Big(\mathfrak{D}(gT; t) \Big[(\pi_1^* \boldsymbol{\omega}_{G/T}) \wedge (\pi_2^* \boldsymbol{\omega}_T) \Big]\Big|_{(eT;e)} \Big). \end{aligned}$$

Multiplying from left by $L^*_{(g;t)}$ we obtain

$$\Big(L_{gt^{-1}g^{-1}} \circ \Psi \circ L_{(g;t)}\Big)^* \boldsymbol{\omega}_G|_e = \mathfrak{D}(gT;t)\Big[(\pi_1^*\boldsymbol{\omega}_{G/T}) \wedge (\pi_2^*\boldsymbol{\omega}_T)\Big]\Big|_{(eT;e)},$$

so after identifying $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$ we have $\mathfrak{D}(gT;t) \equiv \det\Big(L_{gt^{-1}g^{-1}} \circ \Psi \circ L_{(g;t)}\Big)_*$.

Taking any vector at identity $(\boldsymbol{a} \oplus \boldsymbol{b}) \in \mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t} = \mathfrak{g}$ with the corresponding curves $x_\bullet \colon \mathbb{R} \to G/T$ and $y_\bullet \colon \mathbb{R} \to T$, that is $x_{\tau=0} \equiv eT$, $y_{\tau=0} \equiv e$, and

$$\boldsymbol{a} \equiv \frac{\boldsymbol{D}}{d\tau} x_\tau\Big|_{\tau=0} \quad \text{and} \quad \boldsymbol{b} \equiv \frac{\boldsymbol{D}}{d\tau} y_\tau\Big|_{\tau=0},$$

we can calculate

$$\begin{aligned}
\Big(L_{gt^{-1}g^{-1}} \circ \Psi \circ L_{(g;t)}\Big)_*(\boldsymbol{a} \oplus \boldsymbol{b}) &= \frac{\boldsymbol{D}}{d\tau} L_{gt^{-1}g^{-1}} \circ \Psi \circ L_{(g;t)}(x_\tau; y_\tau)\Big|_{\tau=0} \\
\text{just apply } L_{(g;t)} \quad &= \frac{\boldsymbol{D}}{d\tau} L_{gt^{-1}g^{-1}} \circ \Psi(gx_\tau; ty_\tau)\Big|_{\tau=0} \\
\text{apply } \Psi(gT;t) \equiv gtg^{-1} \quad &= \frac{\boldsymbol{D}}{d\tau} L_{gt^{-1}g^{-1}}(gx_\tau)(ty_\tau)(gx_\tau)^{-1}\Big|_{\tau=0} \\
\text{apply } L_{gt^{-1}g^{-1}} \text{ and cancel } gg^{-1} \quad &= \frac{\boldsymbol{D}}{d\tau}\, g\, t^{-1} x_\tau\, t\, y_\tau\, x_\tau^{-1} g^{-1}\Big|_{\tau=0} \\
\text{use } \mathrm{AD}_g h \equiv ghg^{-1} \quad &= \frac{\boldsymbol{D}}{d\tau} \mathrm{AD}_g\Big(\mathrm{AD}_{t^{-1}}(x_\tau)\, y_\tau\, x_\tau^{-1}\Big)\Big|_{\tau=0} \\
\text{use } \mathrm{Ad}_g \equiv \mathrm{AD}_{g*}\big|_e \quad &= \mathrm{Ad}_g \frac{\boldsymbol{D}}{d\tau} \mathrm{AD}_{t^{-1}}(x_\tau)\, y_\tau\, x_\tau^{-1}\Big|_{\tau=0} \\
\text{use Remark 3.76} \quad &= \mathrm{Ad}_g(\mathrm{Ad}_{t^{-1}}\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{a}) \\
\text{matrix notation} \quad &= \mathrm{Ad}_g \left(\begin{array}{c:c} \mathrm{Ad}_{t^{-1}}|_{\mathfrak{g}/\mathfrak{t}} - \mathbf{1}_{\mathfrak{g}/\mathfrak{t}} & \mathbf{0}_{\mathfrak{t}\to\mathfrak{g}/\mathfrak{t}} \\ \hdashline \cdots & \mathbf{1}_{\mathfrak{t}} \end{array}\right)(\boldsymbol{a};\boldsymbol{b})
\end{aligned}$$

Recalling Remark 3.82 for connected G actually leads to the conclusion that $\mathrm{Ad}_\bullet$ is *special orthogonal*, thus $\det \mathrm{Ad}_g = 1$. Overall, we obtain

$$\mathfrak{D}(gT;t) = \det\left(\begin{array}{c:c} \mathrm{Ad}_{t^{-1}}|_{\mathfrak{g}/\mathfrak{t}} - \mathbf{1}_{\mathfrak{g}/\mathfrak{t}} & \mathbf{0}_{\mathfrak{t}\to\mathfrak{g}/\mathfrak{t}} \\ \hdashline \cdots & \mathbf{1}_{\mathfrak{t}} \end{array}\right) = \det{}_{\mathfrak{g}/\mathfrak{t}}(\mathrm{Ad}_{t^{-1}} - \mathbf{1}).$$

Since $\mathfrak{D}$ does not depend on gT, we can write $\mathfrak{D}(t) \equiv \mathfrak{D}(eT;t)$ and calculate

$$\int_{G/T\times T} f\, \Psi^*(\boldsymbol{\omega}_G) = \int_{G/T\times T} f\, \mathfrak{D}\boldsymbol{\omega}_{G/T\times T} = \int_T f\, \mathfrak{D}\, \overbrace{\left(\int_{G/T} \boldsymbol{\omega}_{G/T}\right)}^{1} \boldsymbol{\omega}_T.$$

Proof is thus concluded by utilizing Definition 3.95 and Remark 3.96 to calculate

$$\mathfrak{D}(t) = \det{}_{\mathfrak{g}/\mathfrak{t}}(\mathrm{Ad}_{t^{-1}} - \mathbf{1}) = \prod_{\boldsymbol{\alpha}\in \mathrm{R}(G)} \Big(\boldsymbol{x}^{-\boldsymbol{\alpha}} - 1\Big) = \prod_{\boldsymbol{\alpha}\in \mathrm{R}(G)} (1 - \boldsymbol{x}^{\boldsymbol{\alpha}}),$$

because the invariant integration over the torus T is given by

$$\int_T (\cdots)\, \boldsymbol{\omega}_T = \int\!\!\cdots\!\!\int_0^{2\pi} (\cdots) \prod_{i=1}^{\mathsf{r}} \frac{d\theta_i}{2\pi} \equiv \oint\!\!\cdots\!\!\oint_{|x_i|=1} (\cdots) \prod_{i=1}^{\mathsf{r}} \frac{dx_i}{2\pi i x_i}. \qquad \square$$

Every class function is invariant with respect to the Weyl group $\mathfrak{W}$, thus we also have the simplified form (for derivation see [1, Appendix B])

$$\int_G f(g)\,dg = \oiint\limits_{|x_i|=1} f(\boldsymbol{x}) \underbrace{\left[\prod_{\boldsymbol{\alpha}\in \mathrm{R}_+(G)} (1-\boldsymbol{x}^{\boldsymbol{\alpha}})\right]}_{\mathfrak{D}^+_G(\boldsymbol{x})} \left[\prod_i \frac{dx_i}{2\pi i x_i}\right],$$

where $\mathrm{R}_+(G)$ is the set of so-called *positive roots*.

Example 3.99. Specifically, for $G = \mathsf{SO}(\mathsf{d})$ we have explicit forms [1]

$$\mathfrak{D}^+_{\mathsf{SO}(\mathsf{d})}(\boldsymbol{x}) = \begin{cases} \prod\limits_{1\le i<j\le \mathsf{r}} (1-x_ix_j)(1-x_i/x_j) & \text{for } \mathsf{d}=2\mathsf{r}, \\ \prod\limits_{i=1}^{\mathsf{r}}(1-x_i) \prod\limits_{1\le i<j\le \mathsf{r}} (1-x_ix_j)(1-x_i/x_j) & \text{for } \mathsf{d}=2\mathsf{r}+1. \end{cases}$$

4. The Master Formula

In this chapter we will derive the formula for the Hilbert series by utilizing everything we learned about the deep representation theory of compact Lie groups. At first, we will work in the special setting of a single real scalar (bosonic) field, where the arguments are more transparent and thus easier to follow. Afterwards, we will outline the generalization of the formalism to arbitrary field content.

4.1 Strategy for scalar field

The main strategy for a scalar field Φ follows as:

(1) Construct the *Single Particle Graded Representation* R_Φ (SPGR) containing all possible operators consisting of only one Φ and any number of derivatives (of type $\partial^n\Phi$), excluding ones that are trivially zero by use of EOM.

— addressing EOM relations

(2) Combine multiple R_Φ (consistently with the statistics of Φ) to create the *Multi-Particle Graded Representation* $\mathcal{J}_\Phi$ (MPGR) capturing the highly redundant set of all possible non-trivial operators (of type $\partial^n\Phi^r$).

— representation theory inherently addresses GDC relations

(3) Project out the multiplicities of independent Lorentz and gauge invariant operators from $\mathcal{J}_\Phi$ by use of character orthogonality and formulas for graded characters, thus obtaining the Hilbert series for the operator basis $\mathcal{B}$, already viewed as a graded dimension of $\mathcal{K} \equiv \operatorname{Span}\mathcal{B}$.

— picking out only scalar operators and addressing IBP relations

This can be summarized in the diagram

$$\Phi \xmapsto[\text{and EOM}]{\bigoplus_{n=0}^{\infty}\mathcal{D}^n\partial^n\bullet} R_\Phi \xmapsto[\text{since } \Phi \text{ is a boson}]{\bigoplus_{r=0}^{\infty}\phi^r S^r(\bullet)} \mathcal{J}_\Phi \xmapsto[\text{and Weyl integration formula}]{\int_{\mathsf{SO(d)}}\frac{1}{P(\mathcal{D};g)}\chi_\bullet(\phi,\mathcal{D};g)dg} H_0(\phi,\mathcal{D}),$$

where $H_0(\phi,\mathcal{D})$ is the "main" part (to be defined) of the full Hilbert series $H(\phi,\mathcal{D})$, and the factor $1/P(\mathcal{D};g)$ accounts for the IBP relations.

We will treat steps **(1)** and **(2)** in Section 4.2, afterwards in Section 4.3 we will derive the factor $1/P(\mathcal{D};g)$ appearing in the step **(3)**, and we will finalize the derivation for the scalar field in Section 4.4, with pretty straightforward generalization in Section 4.5.

4.2 Equations of motion redundancy

We would like to build R_Φ by repeatedly applying derivatives on Φ, but we also need to continuously utilize all possible relations to avoid any redundancies. At this point only EOM relations can play a role, so we need to identify them.

The kinetic Lagrangian density of a single scalar field Φ has the form

$$\mathcal{L}_{\text{kin}}(\Phi, \partial_a \Phi) \equiv -\frac{1}{2} \partial_a \Phi \partial^a \Phi \quad \text{in signature}(-,+,+,\dots),$$

which leads to the free equation of motion (known as the *wave equation*)

$$\Box \Phi \equiv \overline{\partial \partial} \Phi \equiv \partial^a \partial_a \Phi = 0.$$

Applying one derivative, we obtain $\partial_a \Phi$, which will be our prototype of an object transforming under the standard representation $\square \equiv \mathbb{C}^{\mathsf{d}}$ of $\mathsf{SO}(\mathsf{d})$. Alternatively we could work with $\partial^a \Phi$, however for $\mathsf{SO}(\mathsf{d})$ this leads to an equivalent representation, since we have invariant metric tensor $\boldsymbol{\delta}_{ab}$ enabling us to lower indices.

When we apply multiple derivatives we obtain symmetric representations $S^n(\square)$, since partial derivatives automatically commute. But additionally, we must not forget to apply EOM, thus we need to discard all *trace* parts of any operator. Since for fully symmetric tensor (representation) there is only one independent contraction, this motivates us to define following representations.

Definition 4.1 (Traceless symmetric representations of $\mathsf{SO}(\mathsf{d})$)**.** Let V be a representation of $\mathsf{SO}(\mathsf{d})$, and $Q(\bullet, \bullet)$ be a corresponding positive-definite bilinear form (see Remark 3.12). We have a natural *contraction* for $n \geq 2$

$$\begin{aligned} C\colon S^n(V) &\to S^{n-2}(V), \\ \boldsymbol{v}_1 \cdots \boldsymbol{v}_n &\mapsto \sum_{i,j} Q(\boldsymbol{v}_i, \boldsymbol{v}_j) \boldsymbol{v}_1 \cdots \widehat{\boldsymbol{v}}_i \cdots \widehat{\boldsymbol{v}}_j \cdots \boldsymbol{v}_n, \end{aligned}$$

which can easily be checked to be a G-map using the G-invariance of $Q(\bullet, \bullet)$. Thus, $\operatorname{Ker}(C)$ is subrepresentation of $S^n(V)$, and it gives us a decomposition

$$S^n(V) = \operatorname{Ker}(C) \oplus \frac{S^n(V)}{\operatorname{Ker}(C)} \cong \operatorname{Ker}(C) \oplus \operatorname{Im}(C) \equiv S^{\{n\}}(V) \oplus S^{n-2}(V),$$

since C is obviously surjective.

Consequently, we have (by definition $S^{\{0\}}(V) \equiv \mathbb{C}$ and $S^{\{1\}}(V) \equiv V$)

$$S^n(V) = S^{\{n\}}(V) \oplus S^{\{n-2\}}(V) \oplus \cdots \oplus S^{\{n-2p\}}(V),$$

where $p = \left\lfloor \frac{d}{2} \right\rfloor$. We call representations $S^{\{n\}}(V)$ as *traceless symmetric*.

Remark 4.2. Similar construction can be made for the tensor power $V^{\otimes n}$, only there we have more contractions, so decomposing it leads to the traceless part and multiple copies of $V^{\otimes n-2}$.

Remark 4.3. Suppose we have a representation W and assume there exists some G-invariant contraction C on W. Then if W wants to be irreducible, it necessarily must be *traceless*, otherwise $\operatorname{Ker}(C)$ would be its nonzero proper subrepresentation (nonzero since it maps to a lower-dimensional space).

Remark 4.4. Representations $S^{\{n\}}(\square)$ for the standard representation $\square \equiv \mathbb{C}^{\mathsf{d}}$ of $\mathsf{SO}(\mathsf{d})$ already turn out to be irreducible [9].

We can create *traceless symmetric graded representation* $S^{\{\bullet\}}(V)$ similarly as $S(V)$. It is in our interest to calculate the corresponding graded character.

Proposition 4.5 (Graded character of traceless symmetric graded representation)**.** Let V be a representation of a compact Lie group G. We define the *traceless symmetric graded representation* $S^{\{\bullet\}}(V)$ of V by

$$S^{\{\bullet\}}(V) \equiv \bigoplus_{n=0}^{\infty} t^n S^{\{n\}}(V).$$

Corresponding graded character $\chi_{S^{\{\bullet\}}(V)}(t;g)$ is given by

$$\chi_{S^{\{\bullet\}}(V)}(t;g) \equiv \sum_{n=0}^{\infty} t^n \chi_{S^{\{n\}}(V)}(g) = (1-t^2)\chi_{S(V)}(t;g) = \frac{1-t^2}{\det_V(1-tg)}.$$

Proof. Using Definition 4.1 we obtain for $n \geq 2$.

$$S^n(V) \equiv S^{\{n\}}(V) \oplus S^{n-2}(V) \overset{3.49}{\Longrightarrow} \chi_{S^n(V)}(g) = \chi_{S^{\{n\}}(V)}(g) + \chi_{S^{n-2}(V)}(g).$$

For convenience we extend the definition of $\chi_{S^n(V)}$ for negative n by zero. We can now simply express $\chi_{S^{\{n\}}(V)}(g)$ and calculate

$$\begin{aligned}\sum_{n=0}^{\infty} t^n \chi_{S^{\{n\}}(V)}(g) &= \sum_{n=0}^{\infty} t^n \big[\chi_{S^n(V)}(g) - \chi_{S^{n-2}(V)}(g)\big] \\ &= (1-t^2)\sum_{n=0}^{\infty} t^n \chi_{S^n(V)}(g) \equiv (1-t^2)\chi_{S(V)}(t;g),\end{aligned}$$

thus the proof is finished by Proposition 3.70. □

Derivation 4.6 (Single Particle Graded Representation R_Φ for a single scalar field Φ)**.** All operators composed of one Φ modulo EOM are therefore contained in

$$R_\Phi = \mathrm{Span}\begin{pmatrix} \Phi \\ \partial_a \Phi \\ \partial_{\{a_1}\partial_{a_2\}}\Phi \\ \vdots \\ \partial_{\{a_1}\cdots\partial_{a_n\}}\Phi \\ \vdots \end{pmatrix} \equiv \bigoplus_{n=0}^{\infty} \mathcal{D}^n S^{\{n\}}(\square) \equiv S^{\{\bullet\}}(\square),$$

where $_{\{\dots\}}$ denotes the traceless symmetric part and $\square \equiv \mathbb{C}^{\mathsf{d}}$ denotes the standard representation of $\mathsf{SO}(\mathsf{d})$. We suggestively changed the grading labels $t \mapsto \mathcal{D}$.

Derivation 4.7 (Multi-Particle Graded Representation $\mathcal{J}_\Phi$ for a single scalar field Φ)**.** Since Φ is a boson, corresponding operators must obey the permutation symmetry.

We can obtain all operators modulo EOM in the symmetric powers of R_Φ, thus

$$\mathcal{J}_\Phi \equiv \bigoplus_{r=0}^{\infty} \phi^r S^r(R_\Phi) \equiv S(R_\Phi) \equiv S\Big(S^{\{\bullet\}}(\square)\Big).$$

Corresponding graded character can be calculated using Corollary 3.72 and Proposition 4.5 as (where we used notation $P(\mathcal{D};g)$ from Example 3.93)

$$\chi_{\mathcal{J}_\Phi}(\phi,\mathcal{D};g) \equiv \chi_{S(R_\Phi)}(\phi,\mathcal{D};g) = \mathrm{PE}\Big[\phi\,\chi_{R_\Phi}(\mathcal{D};g)\Big] = \mathrm{PE}\Big[\phi(1-\mathcal{D}^2)P(\mathcal{D};g)\Big].$$

Remark 4.8 (Gram determinant conditions)**.** We are building $\mathcal{J}_\Phi$ in a certain sense constructively. Since the representation theory intimately understands that representations $\bigwedge^n(\square)$ for $n > \mathsf{d}$ are trivially zero, operators which generate GDC relations (see Section 2.1) are automatically discarded during the process.

Remark 4.9. Alternatively, we can view $\mathcal{J}_\Phi$ as a graded differential polynomial quotient ring with the ideal generated from the equations of motion

$$\begin{aligned}\mathcal{J}_\Phi &= \mathbb{C}[\Phi;\partial_a]\Big/\Big\langle \overset{\square}{\partial\partial}\Phi\Big\rangle \\ &\equiv \mathbb{C}\Big[\Phi, \partial_a\Phi, \partial_{\{a_1}\partial_{a_2\}}\Phi, \dots, \partial_{\{a_1}\cdots\partial_{a_n\}}\Phi, \dots\Big].\end{aligned}$$

4.3 Integration by parts redundancy

This section will be already applicable in general, but we can always imagine we are working with a scalar field Φ.

Suppose we have constructed the redundant representation $\mathcal{J}$ of all possible operators modulo EOM, together with its graded character. Our goal is to find the Hilbert series, which is the graded dimension of $\mathcal{K} \equiv \operatorname{Span}\mathcal{B} \subset \mathcal{J}$. We need to deal with IBP relations and also pick out only scalar (Lorentz and gauge invariant) operators. This will be addressed by a *cohomology*-type calculation.

We would like to proceed similarly as in part **(3)** of Example 2.14, where we realized that any non-trivial operator generates exactly one IBP relation between operators containing one more derivative. Likewise for $\mathsf{d} \geq 2$, operators with one free index can generate IBP relations, but only those that are not trivially zero after applying the divergence. Prime example of operators which do not contribute to the IBP relations are operators of the form

$$\partial^a\mathcal{O}_{ab} \;\text{ where }\; \mathcal{O}_{ab} \equiv \mathcal{O}_{[ab]} \quad\Longrightarrow\quad \partial^b\partial^a\mathcal{O}_{ab} = \partial^{(b}\partial^{a)}\mathcal{O}_{[ab]} = 0,$$

that is so called co-exact 1-forms. Thus, we are naturally led to the following elementary concepts from homology/cohomology/Hodge theory.

Definition 4.10 (Co-closed and co-exact forms)**.** Let $\operatorname{Sect}\boldsymbol{\Lambda}^k\mathcal{M}$ be the space of k-forms on the space-time manifold $\mathcal{M}$. We define the following notation for a form $\boldsymbol{\omega} \in \operatorname{Sect}\boldsymbol{\Lambda}^k\mathcal{M}$ (thus $\boldsymbol{\sigma} \in \operatorname{Sect}\boldsymbol{\Lambda}^{k+1}\mathcal{M}$)

$$\begin{aligned}\boldsymbol{\omega} \text{ is } \textit{co-closed} \quad &\overset{\text{def}}{\Longleftrightarrow} \quad \partial\cdot\boldsymbol{\omega} = 0, \\ \boldsymbol{\omega} \text{ is } \textit{co-exact} \quad &\overset{\text{def}}{\Longleftrightarrow} \quad \exists\,\boldsymbol{\sigma}\colon \boldsymbol{\omega} = \partial\cdot\boldsymbol{\sigma}.\end{aligned}$$

Remark 4.11. For forms we automatically have $\partial \cdot \partial \cdot \bullet = 0$ by symmetry of partial derivatives and asymmetry of forms, thus every co-exact form is also co-closed.

In our case we work with the Euclidean space-time manifold $\mathcal{M} \equiv \mathbb{R}^{\mathsf{d}}$ and the Lorentz group $\mathsf{SO}(\mathsf{d})$, so the k-forms transform under the representation $\bigwedge^k(\square)$, where $\square \equiv \mathbb{C}^{\mathsf{d}}$ is the standard representation of $\mathsf{SO}(\mathsf{d})$.

We would like to identify how many forms of a given degree are contained in $\mathcal{J}$, and use this information to count the operators in $\mathcal{K}$. This motivates us to study $\mathsf{SO}(\mathsf{d})$-linear maps between $\bigwedge^k(\square)$ and $\mathcal{J}$, because images of such non-trivial maps identify subrepresentations of $\mathcal{J}$ with the similar or same transformation properties as have k-forms.

Definition 4.12 (k-form graded representation)**.** Let $\mathcal{J}$ be a multi-particle graded representation of the group $\mathsf{SO}(\mathsf{d})$. The *k-form graded representation* of $\mathcal{J}$ is defined by

$$\mathcal{J}_{[k]} \equiv \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigwedge^k(\square), \mathcal{J}\Big).$$

We say that $\mathcal{O} \in \mathcal{J}$ is a *k-form operator* (we denote it by $\mathcal{O}_{[k]}$) if $\mathcal{O}$ is in the image of some map from $\mathcal{J}_{[k]}$. Furthermore, by straightforward extension of Definition 4.10 to $\mathcal{J}_{[k]}$ we can define its subspaces, for example

$$\mathcal{J}_{[k]\text{co-closed}} \equiv \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigwedge^k(\square), \{\mathcal{O} \in \mathcal{J} \mid \partial \cdot \mathcal{O} = 0\}\Big),$$

$$\mathcal{J}_{[k]\text{co-exact}} \equiv \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigwedge^k(\square), \Big\{\mathcal{O} \in \mathcal{J} \;\Big|\; \exists\, \mathcal{O}'_{[k+1]} \colon \mathcal{O} = \partial \cdot \mathcal{O}'_{[k+1]}\Big\}\Big).$$

To show this definition is indeed sensible, we need to make several comments about the exterior representations of $\mathsf{SO}(\mathsf{d})$.

Remark 4.13. On any pseudo-Riemannian space, but we will specially take $\mathbb{R}^{\mathsf{d}}$, we have the *Hodge star operator* $*\colon \mathbf{\Lambda}^k\mathbb{R}^{\mathsf{d}} \to \mathbf{\Lambda}^{\mathsf{d}-k}\mathbb{R}^{\mathsf{d}}$ defined in components by

$$\boldsymbol{\omega}_{a_1\cdots a_k} \;\overset{*}{\longmapsto}\; (*\boldsymbol{\omega})_{a_{k+1}\cdots a_{\mathsf{d}}} \equiv \frac{1}{k!}\boldsymbol{\omega}^{a_1\cdots a_k}\boldsymbol{\varepsilon}_{a_1\cdots a_k a_{k+1}\cdots a_{\mathsf{d}}},$$

where $\boldsymbol{\varepsilon}$ is so-called *Levi-Civita* tensor.

Since $\boldsymbol{\varepsilon}$ is invariant with respect to $\mathsf{SO}(\mathsf{d})$ transformations, after $\mathbb{C}$-linear extension of Hodge star $*$ to $\square \equiv \mathbb{C}^{\mathsf{d}}$ we get an isomorphism

$$\bigwedge^k(\square) \cong \bigwedge^{\mathsf{d}-k}(\square) \quad\Longrightarrow\quad \mathcal{J}_{[k]} \cong \mathcal{J}_{[\mathsf{d}-k]}.$$

This just reflects representation theory "understands" that any k-form can be transformed by contraction with $\boldsymbol{\varepsilon}$ to $(\mathsf{d}-k)$-form, or vice versa.

Remark 4.14. It is a standard result, that representations $\bigwedge^k(\square)$ are irreducible for $k < \mathsf{d}/2$ [8, 9] (thus also for $k > \mathsf{d}/2$, see Remark 4.13). This is very important, because for such k we can use Corollary 3.50, or maybe even better Remark 3.68.

But then there is the case $k = \mathsf{d}/2$ for even d, which needs more justification. Hodge star operator satisfies the identity

$$* \circ * = (-1)^{k(\mathsf{d}-k)}\mathbf{1},$$

so for $\mathsf{d} = 2m$ we obtain the automorphism of kth exterior representation of $\square$

$$\tau \equiv (-i)^m *\colon \bigwedge^m(\square) \to \bigwedge^m(\square), \quad \tau \circ \tau = \mathbf{1},$$

giving us the canonical decomposition to *self-dual* and *anti self-dual* representations

$$\bigwedge^m(\square) = \bigwedge^m_+(\square) \oplus \bigwedge^m_-(\square),$$

which are eigenspaces of τ corresponding to the eigenvalues 1 and -1, respectively. These already turn out to be irreducible (and mutually inequivalent), therefore

$$\dim \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigwedge^m(\square), \bigwedge^m(\square)\Big) = \sum_{\pm} \dim \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigwedge^m(\square), \bigwedge^m_{\pm}(\square)\Big) = 1 + 1 = 2.$$

This is exactly what we want, because each of the 2 independent $m = \mathsf{d}/2$ form representations plays their own role in the following derivation.

Derivation 4.15 (Adressing IBP relations by cohomology, splitting $H = H_0 + \Delta H$)**.** Since IBP relations tell us that total divergence terms are equivalent to zero, the representation $\mathcal{K} \equiv \mathrm{Span}(\mathcal{B})$ is composed of all 0-forms contained in $\mathcal{J}$ that are not co-exact. In other words, the Hilbert series is given by the graded multiplicities of trivial representations in $\mathcal{J}$ that are not co-exact, that is

$$H(\boldsymbol{\phi}, \mathcal{D}) \equiv \dim_{(\boldsymbol{\phi},\mathcal{D})} \mathcal{K} = \dim_{(\boldsymbol{\phi},\mathcal{D})} \mathcal{J}_{[0]\text{not co-exact}}.$$

For the sake of brevity, we will write just $\dim \bullet$ instead of $\dim_{(\boldsymbol{\phi},\mathcal{D})} \bullet$. Since to every non-trivial co-exact k-form corresponds exactly one $(k+1)$-form that is not co-closed (see Definition 4.10), by keeping track of the right grading we obtain

$$\begin{aligned}
\dim \mathcal{K} &= \overbrace{\dim \mathcal{J}_{[0]} - \quad \dim \mathcal{J}_{[0]\text{co-exact}}}^{\dim \mathcal{J}_{[0]\text{not co-exact}}} \\
&= \dim \mathcal{J}_{[0]} - \mathcal{D}\ \dim \mathcal{J}_{[1]\text{not co-closed}} \\
&= \dim \mathcal{J}_{[0]} - \mathcal{D}\Big(\dim \mathcal{J}_{[1]} - \dim \mathcal{J}_{[1]\text{co-closed}}\Big) \\
&= \dim \mathcal{J}_{[0]} - \mathcal{D}\Bigg(\dim \mathcal{J}_{[1]} - \underbrace{\dim \mathcal{J}_{[1]\text{co-exact}}}_{\mathcal{D}\dim \mathcal{J}_{[2]\text{not co-closed}}} - \dim \mathcal{J}_{[1]_{\text{not co-exact}}^{\text{co-closed}}}\Bigg) \\
&\ \ \vdots \quad \text{iteratively} \\
&= \underbrace{\sum_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \dim \mathcal{J}_{[k]}}_{H_0} + \underbrace{\sum_{k=1}^{\mathsf{d}} (-1)^{k+1} \mathcal{D}^k \dim \mathcal{J}_{[k]_{\text{not co-exact}}^{\text{co-closed}}}}_{\Delta H},
\end{aligned}$$

where we also used splitting of co-closed forms to the co-exact ones and to the rest (see Remark 4.11). Iteration terminates because there are no k-forms with $k > \mathsf{d}$.

Hence, we found that the Hilbert series naturally splits into two pieces, where we will intuitively infer in Remark 4.16 that ΔH is a small correction to the "main" part H_0. Since H_0 usually contains almost the whole information of H, we sometimes refer to it also as the Hilbert series. We will obtain an exact formula for H_0, but calculation of ΔH must be done "by hand".

Remark 4.16 (Contributions to ΔH)**.** One such co-closed but not co-exact form is the Levi-Civita tensor $\boldsymbol{\varepsilon}_{\bullet\dots\bullet} \equiv *1$ (where $*$ denotes the Hodge dual). This follows because it is constant, thus co-closed, and also a top-form, so there is no form of higher degree. We therefore always have a contribution of $-(-\mathcal{D})^{\mathsf{d}}$ to ΔH, because we over-count $\boldsymbol{\varepsilon}_{\bullet\dots\bullet}$ in the H_0 part.

In the case of a single scalar field Φ, the one-form $\partial_a\Phi$ also falls down to that category. Co-closedness results from EOM, as $\partial \cdot \partial\Phi \equiv \partial^a\partial_a\Phi = 0$, and it is obvious that it is not co-exact. Since it contains one derivative, and we need another one to bring it down to a 0-form, we obtain a contribution of $\phi\mathcal{D}^2$ to ΔH.

One could try to find other examples in $\mathcal{J}_{[k]\substack{\text{co-closed}\\\text{not co-exact}}}$, but in the case of a scalar field there are none. That such terms are pretty rare sounds plausible. If the divergence of a form is not automatically zero by its components' symmetry, only other possibilities are that it is either constant (only $\boldsymbol{\varepsilon}_{\bullet\dots\bullet}$) or it vanishes by EOM. This typically leads to only couple of instances, since the divergence of a form containing couple of particle fields is unlikely to be identically zero (for example $\partial^a(\partial_a\Phi\Phi) = \partial^a\partial_a\Phi\Phi + \partial_a\Phi\partial^a\Phi \neq 0$), and if we have to use only one particle field, there are not many forms because $\partial_{[\bullet}\partial_{\bullet]} \equiv 0$.

We conclude that for a single scalar field we have

$$\Delta H(\phi, \mathcal{D}) = (-1)^{\mathsf{d}+1}\mathcal{D}^{\mathsf{d}} + \phi\mathcal{D}^2,$$

which agrees with an explicit formula for ΔH derived by Henning, Lu, Melia, and Murayama [1], which is valid only in some special cases when SPGR corresponds to a *conformal* representation (in particular for a scalar field).

4.4 Integral formula for the Hilbert series

In this section we finalize the derivation of formula for the Hilbert series, namely for the "main" part H_0. We essentially have done the hard work in Section 4.3, now we just (in the same spirit as in Corollary 3.50) express the graded dimension occurring in H_0 through integration of the graded characters.

Derivation 4.17 (Projection factor $1/P(\mathcal{D}; g)$ addressing IBP relations)**.** First, we can perform following manipulations (just to obtain nice alternative expression)

$$\begin{aligned} H_0(\boldsymbol{\phi}, \mathcal{D}) &\equiv \sum_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \dim_{(\boldsymbol{\phi},\mathcal{D})} \mathcal{J}_{[k]} \equiv \sum_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \dim_{(\boldsymbol{\phi},\mathcal{D})} \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigwedge\nolimits^k(\square), \mathcal{J}\Big) \\ &= \dim_{(\boldsymbol{\phi},\mathcal{D})} \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigoplus_{k=0}^{\mathsf{d}} (-\mathcal{D})^k \bigwedge\nolimits^k(\square), \mathcal{J}\Big) \equiv \dim_{(\boldsymbol{\phi},\mathcal{D})} \mathrm{Hom}_{\mathsf{SO}(\mathsf{d})}\Big(\bigwedge\nolimits^-(\square), \mathcal{J}\Big), \end{aligned}$$

where $\bigwedge^-(\square)$ is the exterior graded representation of $\square$, but with alternating signs in the grading. Using Proposition 3.70 we can calculate its graded character as

$$\chi_{\bigwedge^-(\square)}(\mathcal{D}; g) \equiv \sum_{n=0}^{\infty} (-\mathcal{D})^n \chi_{\bigwedge^n(\square)}(g) = \det{}_{\square}(1 - \mathcal{D}g) = \frac{1}{\chi_{S(\square)}(\mathcal{D}; g)} \equiv \frac{1}{P(\mathcal{D}; g)}.$$

Derivation 4.18 (Master Formula for H_0)**.** Since Derivation 4.15 was not specific to a single scalar field, we quite generally obtain *the Master Formula* for H_0 as

$$H_0(\boldsymbol{\phi},\mathcal{D}) = \dim_{(\boldsymbol{\phi},\mathcal{D})} \mathrm{Hom}_{\mathsf{SO(d)}}\Big(\bigwedge\nolimits^{-}(\square), \mathcal{J}\Big) = \int_{\mathsf{SO(d)}} \chi_{\bigwedge^{-}(\square)}(\mathcal{D}; g^{-1})\chi_{\mathcal{J}}(\boldsymbol{\phi},\mathcal{D};g)\,dg$$
$$= \int_{\mathsf{SO(d)}} \underbrace{\det{}_{\square}(1-\mathcal{D}g^{\intercal})}_{\det_{\square}(1-\mathcal{D}g)} \chi_{\mathcal{J}}(\boldsymbol{\phi},\mathcal{D};g)\,dg \equiv \int_{\mathsf{SO(d)}} \frac{1}{P(\mathcal{D};g)}\chi_{\mathcal{J}}(\boldsymbol{\phi},\mathcal{D};g)\,dg\,,$$

where we used equality **(2)** of Corollary 3.50 extended to the case of graded representations, together with relation $g^{-1} = g^{\intercal}$ valid for $\mathsf{SO(d)}$ (see Example 3.10). Integration can be further simplified by restricting it to the torus T of $\mathsf{SO(d)}$ (using the Weyl integration formula discussed in Section 3.6).

For the special case of a single scalar field we therefore obtain

$$H(\phi,\mathcal{D}) = \underbrace{\int_{\mathsf{SO(d)}} \frac{1}{P(\mathcal{D};g)}\,\mathrm{PE}\Big[\phi(1-\mathcal{D}^2)P(\mathcal{D};g)\Big]\,dg}_{H_0(\phi,\mathcal{D})} + \underbrace{(-1)^{\mathsf{d}+1}\mathcal{D}^{\mathsf{d}} - \phi\mathcal{D}^2}_{\Delta H(\phi,\mathcal{D})}\,.$$

We will continue the calculation in Section 5.1.

4.5 Generalization to arbitrary field content

We are *almost* ready to generalize the derivation to the arbitrary field content. To say *almost* is actually a big overstatement, since there are numerous places where it is necessary to use sophisticated tools far beyond the scope of this thesis. Nonetheless, the big picture can be understood very well after the worked out case of a single scalar field.

Our starting data for any EFT are:

- Particle fields $\{\Phi_i\}$ together with specification of their representations under the Lorentz group $\mathsf{SO(d)}$ and a possible internal group G composed of gauge groups and other (for example global) symmetry groups. We always assume particle fields transform linearly under G.
- EOMs generated from the kinetic Lagrangian density $\mathcal{L}_{\mathrm{kin}}$.
- Other constraints, example being Bianchi identities for Maxwell tensor

$$3\partial_{[a}F_{bc]} = \mathrm{d}_a F_{bc} = 0.$$

We follow the strategy in Section 4.1 with slight modifications, see diagram below. Additional internal group G is accounted simply by assigning to every operator its corresponding character with respect to G. Their only role will be played out at the end, where we just project out the scalar operators by integration over G.

$$\{\Phi_i\} \xmapsto[\text{and EOM}]{\bigoplus_{n=0}^{\infty}\mathcal{D}^n\partial^n\bullet} \{R_{\Phi_i}\} \xmapsto[\Phi_i \text{ is a } \substack{\text{boson}\\ \text{fermion}}]{\bigotimes_i\left(\bigoplus_{r=0}^{\infty}\phi_i^r\,S^r(\bullet_i)\right)\left(\bigwedge\nolimits^{r}(\bullet_i)\right)} \mathcal{J} \xmapsto[\text{and Weyl integration formula}]{\int_{\mathsf{SO(d)}\times G}\frac{1}{P(\mathcal{D};g)}\chi_{\bullet}(\phi,\mathcal{D};g)dg} H_0(\boldsymbol{\phi},\mathcal{D})$$

Remark 4.19. Now that we work with the direct product of groups $\mathsf{SO(d)} \times G$ (see Definition 3.7), we will understand group elements as $g \equiv (g_{\mathsf{SO(d)}}; g_G)$. In the following we will just write dependence on g, but it is easy to recognize that some parts of expressions depend only on $g_{\mathsf{SO(d)}}$, others only on g_G.

Remark 4.20 (Fermions)**.** We need to proceed more carefully with particle fields that are fermions (spinors), since objects transforming spinorially must be integrated over the covering group $\mathsf{Spin(d)}$ of $\mathsf{SO(d)}$. This leads to some technicalities, so we will focus mainly on bosons. We refer interested reader to [1, 13].

Derivation 4.21 (Single Particle Graded Representations $\{R_{\Phi_i}\}$ for general fields)**.** Once again, we start by building the single particle graded representations $\{R_{\Phi_i}\}$. In the case of a scalar field it was useful to decompose representations obtained just by applying derivatives to smaller pieces, since some of them were trivially zero by use of EOM. *Decomposition to irreducible representations* under the action of $\mathsf{SO(d)}$ is effective in general, because any constraint (EOM or other) always discards one or several whole irreducible representations. More complicated example of electromagnetic field (photons) will be showed in Section 5.3.

If we successfully apply all EOM and other relations, we will be left with $\{R_{\Phi_i}\}$, which decompose to the space-time $\mathsf{SO(d)}$ part and the internal G part as

$$R_{\Phi_i} = R_{\mathsf{SO(d)},\Phi_i} \otimes R_{G,\Phi_i}.$$

The corresponding graded characters are then

$$\chi_{R_{\Phi_i}}(\mathcal{D}; g) \equiv \chi_{R_{\mathsf{SO(d)},\Phi_i} \otimes R_{G,\Phi_i}}(\mathcal{D}; g) = \chi_{R_{\mathsf{SO(d)},\Phi_i}}(\mathcal{D}; g)\chi_{G,\Phi_i}(g),$$

where χ_{G,Φ_i} is just character of the internal group G representation of the particle field Φ_i, since all operators in R_{Φ_i} contain only one Φ_i.

Remark 4.22 (Young diagrams)**.** Tensor powers $\square^{\otimes n}$ of the standard representation of $\mathsf{SO(d)}$ are typically decomposed by usage of so-called *Young symmetrizers* [11]. The idea it to exploit mutually commuting actions of $\mathsf{SO(d)}$ and the *symmetric* group $\mathfrak{S}_n$ (group of permutations on n symbols), which acts by permuting the indices of the tensor. We can thus decompose $\square^{\otimes n}$ to subrepresentations already irreducible under $\mathfrak{S}_n$, particular examples already showed in Definition 3.55, where $S^n(\square)$ corresponds to the trivial representation, and $\bigwedge^n(\square)$ corresponds to the *sign* representation. Other irreducible representations have "mixed" symmetries. Any irreducible representation of $\mathfrak{S}_n$ is uniquely described by a *Young diagram* with n boxes, that will be intuitively used in Section 5.3.

Derivation 4.23 (Multi-Particle Graded Representation $\mathcal{J}$ for general fields)**.** For Φ_i boson/fermion, we obtain all operators containing only this particle field in symmetric/exterior powers of R_Φ, thus $\mathcal{J}_{\Phi_i}$ is

$$\mathcal{J}_{\Phi_i} \equiv \bigoplus_{r=0}^{\infty} \phi_i^r \begin{smallmatrix} S^r\left(R_{\Phi_i}\right) \\ \bigwedge^r\left(R_{\Phi_i}\right) \end{smallmatrix} = \begin{smallmatrix} S\left(R_{\Phi_i}\right) \\ \bigwedge\left(R_{\Phi_i}\right) \end{smallmatrix}.$$

The full *Multi-Particle Graded Representation* $\mathcal{J}$ containing all possible operators (modulo EOM and other relations) is obtained just by taking the tensor product

$$\mathcal{J} \equiv \bigotimes_i \mathcal{J}_{\Phi_i} = \bigotimes_i \begin{smallmatrix} S\left(R_{\Phi_i}\right) \\ \bigwedge\left(R_{\Phi_i}\right) \end{smallmatrix}.$$

The corresponding graded character is given by Corollary 3.72 as

$$\chi_{\mathcal{J}}(\boldsymbol{\phi},\mathcal{D};g)=\prod_i \chi_{\mathcal{J}_{\Phi_i}}(\phi_i,\mathcal{D};g)=\prod_i \mathrm{PE}\Big[\phi_i\chi_{R_{\Phi_i}}(\mathcal{D};g)\Big]=\mathrm{PE}\Big[\sum_i \phi_i\chi_{R_{\Phi_i}}(\mathcal{D};g)\Big],$$

with the proper statistics understood in PE (see Remark 3.73).

Derivation 4.24 (General Master Formula for the Hilbert series)**.** Finally, by the same arguments as in Section 4.3, only with additional integration over the internal group G, we state *the Master Formula*

$$H(\boldsymbol{\phi},\mathcal{D})=\underbrace{\int_{\mathsf{SO}(\mathsf{d})\times G}\frac{1}{P(\mathcal{D};g)}\chi_{\mathcal{J}}(\boldsymbol{\phi},\mathcal{D};g)\,dg}_{H_0(\boldsymbol{\phi},\mathcal{D})}+\Delta H(\boldsymbol{\phi},\mathcal{D}),$$

where ΔH corrects for miscalculations in H_0 caused by existence of co-closed but not co-exact forms in $\mathcal{J}$, namely

$$\begin{aligned}\Delta H(\boldsymbol{\phi},\mathcal{D})&=\sum_{k=1}^{\mathsf{d}}(-1)^{k+1}\mathcal{D}^k\dim_{(\boldsymbol{\phi},\mathcal{D})}\mathcal{J}_{[k]_{\substack{\text{co-closed}\\ \text{not co-exact}}}}\\&\equiv\sum_{k=1}^{\mathsf{d}}(-1)^{k+1}\mathcal{D}^k\dim_{(\boldsymbol{\phi},\mathcal{D})}\mathrm{Hom}_{\mathsf{SO}(\mathsf{d})\times G}\Big(\bigwedge\nolimits^k(\square),\mathcal{J}_{\substack{\text{co-closed}\\ \text{not co-exact}}}\Big).\end{aligned}$$

In the next chapter we will apply the formalism to some simple examples. Hopefully, they will suffice to unveil the main techniques and solve some ambiguities of our exposition up to this point. Of course, there is much more to learn, and we will try to refer the interested reader to the relevant sources.

5. Applications of the Formalism

We are now fully ready to employ the formalism developed in preceding chapters. We will not be able to obtain the full *Hilbert series*, but by expanding the Plethystic Exponentials we will focus on parts with a fixed field content (see Remark 2.13).

In general, the final integrals we will obtain are just, to put it simply, *gigantic*. Nevertheless, the results can be obtained by "just" calculating a finite number of residues, since the integrands are rational functions. This can be automatized by use of *computer algebra systems*, such as `Mathematica`. Since it is not possible to show all the details of following computations, we encourage the reader to go trough the accompanying `Mathematica notebook` [14].

First, we will finish the calculation for a single scalar field. Analysis of the results will give us numerous enlightening inputs on the structure of the operator basis. We will give a brief discussion regarding the dependence of the Hilbert series on the dimension of the space-time d. After that, we will show on a little more complicated example of electromagnetic field (photons) how to proceed, if the building block of the Lagrangian has non-trivial transformation properties.

5.1 Single scalar field

For concreteness, we will write out equations only for $\mathsf{d} = 4$. Picking up where we left off the case of a single scalar field in Section 4.4, the usage of the Weyl integration formula for $\mathsf{SO}(4)$ (see Theorem 3.97 and Example 3.99) leads to

$$\begin{aligned} H_0(\phi,\mathcal{D}) &= \int_{\mathsf{SO}(4)} \frac{1}{P(\mathcal{D};g)} \overbrace{\mathrm{PE}\Big[\phi(1-\mathcal{D}^2)P(\mathcal{D};g)\Big]}^{\chi_{\mathcal{J}_\Phi}(\phi,\mathcal{D};g)} dg \\ &= \oiint\limits_{\substack{|x_1|=1\\|x_2|=1}} (1-\mathcal{D}x_1)(1-\mathcal{D}/x_1)(1-\mathcal{D}x_2)(1-\mathcal{D}/x_2)\times \\ &\qquad \times \mathrm{PE}\left[\frac{\phi(1-\mathcal{D}^2)}{(1-\mathcal{D}x_1)(1-\mathcal{D}/x_1)(1-\mathcal{D}x_2)(1-\mathcal{D}/x_2)}\right]\times \\ &\qquad\qquad \times(1-x_1x_2)(1-x_1/x_2)\frac{dx_1}{2\pi i x_1}\frac{dx_2}{2\pi i x_2}, \end{aligned}$$

where we have $|\mathcal{D}|<1$ (see Definition 2.9), $\boldsymbol{x}=(x_1,x_2)$ parametrizes the torus T of $\mathsf{SO}(4)$, and we used Example 3.93 to obtain

$$P(\mathcal{D};\boldsymbol{x}) \equiv \chi_{S(\square)}(\mathcal{D};\boldsymbol{x}) = \frac{1}{(1-\mathcal{D}x_1)(1-\mathcal{D}/x_1)(1-\mathcal{D}x_2)(1-\mathcal{D}/x_2)}.$$

Calculation 5.1 (Fixed field content $\Phi^r, \boldsymbol{r}\leq\mathbf{3}$)**.** It is straightforward exercise to show that the only non-trivial operators with the field content $\Phi^{\boldsymbol{r}}, \boldsymbol{r}=\mathbf{1},\mathbf{2},\mathbf{3}$, are exactly $\Phi^{\mathbf{1}}$, $\Phi^{\mathbf{2}}$, and $\Phi^{\mathbf{3}}$ (we use bold numbers only for visibility). This indeed checks out with calculation through the Hilbert series formalism for any given d, see the accompanying `Mathematica notebook`. We must not forget the term

$$\Delta H(\phi,\mathcal{D}) = (-1)^{\mathsf{d}+1}\mathcal{D}^{\mathsf{d}} + \phi\mathcal{D}^2$$

for $\boldsymbol{r}=\mathbf{1}$, otherwise it does not contribute.

Calculation 5.2 (Fixed field content $\Phi^{\mathbf{4}}$)**.** The simplest non-trivial case are operators with $\boldsymbol{r} = \mathbf{4}$ fields Φ. Just by expanding the Plethystic Exponential defined in Remark 3.73 we get ($\boldsymbol{t}$ stands for any set of parameters, and the notation $|_{\phi=4}$ means the coefficient in front of ϕ^4)

$$\mathrm{PE}[\phi f(\boldsymbol{t})]\Big|_{\phi^4} = \frac{1}{24}\Big(f(\boldsymbol{t})^4 + 6f(\boldsymbol{t})^2 f(\boldsymbol{t}^2) + 3f(\boldsymbol{t}^2)^2 + 8f(\boldsymbol{t})f(\boldsymbol{t}^3) + 6f(\boldsymbol{t}^4)\Big),$$

where in the place of $f(\boldsymbol{t})$ we substitute $(1-\mathcal{D}^2)P(\mathcal{D};\boldsymbol{x})$. For example, the term $f(\boldsymbol{t}^4)$ corresponds to ($\boldsymbol{t}^4$ means we take the 4th power of every argument)

$$(1-\mathcal{D}^8)P(\mathcal{D}^4;\boldsymbol{x}^4) = \frac{(1-\mathcal{D}^8)x_1^4x_2^4}{(1-\mathcal{D}^4x_1^4)(x_1^4-\mathcal{D}^4)(1-\mathcal{D}^4x_2^4)(x_2^4-\mathcal{D}^4)}.$$

Just to carry out the calculation a little further, we multiply this term by the rest of the integrand, after which using the residue theorem twice and performing considerable amount of algebra (not included here) gives us

$$\begin{aligned}\int_{\mathsf{SO}(4)} \frac{1}{P(\mathcal{D};g)}\frac{1}{4}(1-\mathcal{D}^8)P(\mathcal{D}^4;g^4)\,dg &= \frac{1}{4}(1-\mathcal{D}^8)\oint\limits_{|x_i|=1}\frac{dx_1}{2\pi i}\frac{dx_2}{2\pi i}\times\\ &\times\frac{(1-x_1x_2)(x_2-x_1)(1-\mathcal{D}x_1)(x_1-\mathcal{D})(1-\mathcal{D}x_2)(x_2-\mathcal{D})x_1^2x_2}{(1-\mathcal{D}^4x_1^4)(x_1^4-\mathcal{D}^4)(1-\mathcal{D}^4x_2^4)(x_2^4-\mathcal{D}^4)}\\ &= \frac{1}{4}\oint\limits_{|x_2|=1}\frac{dx_2}{2\pi i}\frac{x_2}{(1+\mathcal{D}^2x_2^2)(x_2^2+\mathcal{D}^2)} = \frac{1}{4}\frac{1}{1-\mathcal{D}^4}\end{aligned}$$

Poles in the first integration are ($\overset{\text{removable}}{\mathcal{D}}, -\mathcal{D}, i\mathcal{D}, -i\mathcal{D}$), in the second ($i\mathcal{D}, -i\mathcal{D}$).

If we wanted to continue the whole calculation by hand, we would probably lose that hand. Expanding the Plethystic Exponential gave us 5 terms, each leading to a pretty large integrand. Poles are easy to find, but either we have poles of higher degree, so we must take derivatives to calculate the residues, or there are multiple (up to 3) poles for a single term. And we must do that twice, for x_1 and x_2.

Thankfully, `Mathematica` comes to the rescue. *Simply* by calculating the indicated residues and summing them all, we obtain (since $\Delta H|_{\phi^4} = 0$)

$$H_{\mathbf{4}}(\mathcal{D}) \equiv H(\phi,\mathcal{D})\Big|_{\phi^4} = \frac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)} = 1+\mathcal{D}^4+\mathcal{D}^6+\mathcal{D}^8+\mathcal{D}^{10}+2\mathcal{D}^{12}+\cdots,$$

reproducing the result calculated in [1]. This implies there is only one independent operator of the type $\partial^0\Phi^{\mathbf{4}} \equiv \Phi^{\mathbf{4}}$ (obviously), one of the type $\partial^4\Phi^{\mathbf{4}}$, one of the type $\partial^6\Phi^{\mathbf{4}}$, but two of the type $\partial^{12}\Phi^{\mathbf{4}}$, and so on.

Remark 5.3 (Interpretation of the Hilbert series)**.** The interpretation of this Hilbert series is slightly different from the ones in Example 2.14, to which Remark 2.12 is directly applicable. Since now we have fixed field content, the form of $H_{\mathbf{4}}$ reflects independent possibilities of assigning a given number of derivatives to the $\mathbf{4}$ fields Φ and contracting them.

For example, we can take following assignments of 4 and 6 derivatives

$$\overbrace{\partial\partial\Phi\partial\Phi}\partial\Phi\Phi \equiv \partial^b\partial_a\Phi\partial^a\Phi\partial_b\Phi\Phi \quad\text{and}\quad \overbrace{\partial\partial\Phi\partial\overbrace{\Phi\partial}\partial}\Phi\Phi \equiv \partial^c\partial_a\Phi\partial^a\partial_b\Phi\partial^b\partial_c\Phi\Phi$$

to be *"generators"*, and any operator of the form $\partial^n\Phi^{\mathbf{4}}$ can be already constructed by repeatedly assigning derivatives and contracting them in the same way.

Remark 5.4 (Challenge)**.** It is pretty easy to transform any operator of the form $\partial^4\Phi^4$ to the trivial zero operator or to the one stated above, for example

$$\begin{aligned}\overbrace{\partial\Phi\partial}\Phi\overbrace{\partial\Phi\partial}\Phi \equiv \partial_a\Phi\partial^a\Phi\partial_b\Phi\partial^b\Phi &= \overset{\sim 0 \text{ by IBP}}{\partial^b(\partial_a\Phi\partial^a\Phi\partial_b\Phi\Phi)} - \overset{\sim 0 \text{ by EOM}}{\partial_a\Phi\partial^a\Phi\big(\partial^b\partial_b\Phi\big)\Phi} \\ &\quad - \partial^b\partial_a\Phi\partial^a\Phi\partial_b\Phi\Phi - \partial_a\Phi\partial^b\partial^a\Phi\partial_b\Phi\Phi \\ &= -2\partial^b\partial_a\Phi\partial^a\Phi\partial_b\Phi\Phi \equiv -2\overbrace{\partial\partial\Phi\partial\Phi\partial}\Phi\Phi.\end{aligned}$$

We encourage the reader to show that any (non-trivial) operator of the form $\partial^6\Phi^{\mathbf{4}}$ is equivalent to the one stated above. If it does not feel like a challenge, have a go at operators of the form $\partial^8\Phi^{\mathbf{4}}$ or $\partial^{10}\Phi^{\mathbf{4}}$. Hopefully, it will be sufficient evidence, that even in the simplest example of a single scalar field, the combination of EOM and IBP relations leads to complexity that should not be underrated.

And everything gets exponentially more tangled for higher number of fields, where GDC relations start to kick in. Still, the Hilbert series formalism can give us (if we are able to perform the integration) information to all orders in powers of derivatives for any fixed field content $\Phi^{\boldsymbol{r}}$.

The authors of [1] explicitly calculated the Hilbert series of a single scalar field $H_{\boldsymbol{r}}(\mathcal{D})$ for $\boldsymbol{r} = \mathbf{1}, \ldots, \mathbf{8}$, but only in the dimension $\mathsf{d} = 4$. As is expected, our calculations agree with theirs. It is very hard to go beyond the field content of $\Phi^{\mathbf{8}}$ (integrands become so lengthy, that even `Mathematica` has really rough time), so we will rather explore the dependency of the Hilbert series on the dimension d.

Below, in Tables 5.1 and 5.2 we list the Hilbert series for $\boldsymbol{r} = \mathbf{4}, \mathbf{5}, \mathbf{6}$ and all dimensions $\mathsf{d} \geq 2$. For illustration, we list the corresponding expansions to the order of $\partial^{20}\Phi^{\boldsymbol{r}}$ in Tables 5.3 to 5.5. We will make a comment in Section 5.2 concerning the independence of $H_{\boldsymbol{r}}(\mathcal{D})$ on d for $\mathsf{d} \geq \boldsymbol{r}$. For easier comparison of their dependence on d we also include Figures 5.1 and 5.2. To our knowledge, results of $H_{\boldsymbol{r}}(\mathcal{D})$ for $\boldsymbol{r} = \mathbf{5}, \mathbf{6}$ in dimensions other than $\mathsf{d} = 4$ are *novel*.

Remark 5.5 (Analysis of $H_{\mathbf{4}}(\mathcal{D})$)**.** Comparing $H_{\mathbf{4}}(\mathcal{D})$ for $\mathsf{d} = 3$ and $\mathsf{d} = 4$ we can see, that in $\mathsf{d} = 3$ we have one additional operator for every one in $\mathsf{d} = 4$, but with 9 more derivatives. This corresponds to the operator

$$\boldsymbol{\varepsilon}^{abc}(\overbrace{\partial\partial\partial_a\Phi)(\partial\partial\partial_b\Phi)(\partial\partial_c\Phi)}(\Phi).$$

Such assignment of derivatives "can be used only once", because by repeating it twice, we can deconstruct two $\boldsymbol{\varepsilon}^{\bullet\cdots\bullet}$ as a combination of $\boldsymbol{\delta}^{\bullet\bullet}$, thus only contractions, which are already accounted by the denominator.

Why there is no such operator in $\mathsf{d} = 4$, where we would contract derivatives with $\boldsymbol{\varepsilon}^{abcd}$? Since derivatives commute, in order to obtain non-trivial operator, we must contract $\boldsymbol{\varepsilon}^{abcd}$ with derivatives standing before different fields Φ. But we are always able to move all derivatives in front of one Φ to the others by use of IBP relations, thus such operator does not exist. This applies in general for scalar fields, so operators with $\boldsymbol{\varepsilon}^{\bullet\cdots\bullet}$ exist only for $\mathsf{d} < r$. Note that the only source of operators with odd number of derivatives are $\boldsymbol{\varepsilon}^{\bullet\cdots\bullet}$ in odd dimensions.

For some reason, by going down to $\mathsf{d} = 2$ we totally lose the operator of the type $\partial^6\Phi^{\mathbf{4}}$ (and also its "iterations"). We will study this in more detail in Section 5.2.

d	$H_{\mathbf{4}}(\mathcal{D})$	$H_{\mathbf{5}}(\mathcal{D})$
≥ 5		$\dfrac{1+\mathcal{D}^{12}+\mathcal{D}^{14}+\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
4	$\dfrac{1}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$\dfrac{1+\mathcal{D}^{10}+\mathcal{D}^{12}+2\mathcal{D}^{14}+2\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{22}+\mathcal{D}^{24}+\mathcal{D}^{28}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
3	$\dfrac{1+\mathcal{D}^9}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$\dfrac{1+\mathcal{D}^9+\mathcal{D}^{12}+\mathcal{D}^{13}+\mathcal{D}^{14}+2\mathcal{D}^{15}+\mathcal{D}^{16}+\mathcal{D}^{17}+\mathcal{D}^{18}+\mathcal{D}^{21}+\mathcal{D}^{30}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
2	$\dfrac{1}{1-\mathcal{D}^4}$	$\dfrac{1+\mathcal{D}^{12}}{(1-\mathcal{D}^4)(1-\mathcal{D}^{12})}$

Table 5.1: The Hilbert series for a singe scalar field, with a fixed field content $\Phi^{\mathbf{4}}$ and $\Phi^{\mathbf{5}}$.

d	$H_{\mathbf{6}}(\mathcal{D})$
≥ 6	$\dfrac{\begin{array}{c}1+2\mathcal{D}^{10}+5\mathcal{D}^{12}+7\mathcal{D}^{14}+9\mathcal{D}^{16}+11\mathcal{D}^{18}+13\mathcal{D}^{20}+14\mathcal{D}^{22}+21\mathcal{D}^{24}+24\mathcal{D}^{26}\\ +28\mathcal{D}^{28}+32\mathcal{D}^{30}+26\mathcal{D}^{32}+22\mathcal{D}^{34}+13\mathcal{D}^{36}+7\mathcal{D}^{38}+3\mathcal{D}^{40}+\mathcal{D}^{42}+\mathcal{D}^{44}\end{array}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)^3(1-\mathcal{D}^{10})^2(1-\mathcal{D}^{12})}$
5	$\dfrac{\begin{array}{c}1+2\mathcal{D}^{10}+5\mathcal{D}^{12}+7\mathcal{D}^{14}+\mathcal{D}^{15}+9\mathcal{D}^{16}+\mathcal{D}^{17}+11\mathcal{D}^{18}+3\mathcal{D}^{19}+13\mathcal{D}^{20}+7\mathcal{D}^{21}+14\mathcal{D}^{22}+13\mathcal{D}^{23}+21\mathcal{D}^{24}\\ +22\mathcal{D}^{25}+24\mathcal{D}^{26}+26\mathcal{D}^{27}+28\mathcal{D}^{28}+32\mathcal{D}^{29}+32\mathcal{D}^{30}+28\mathcal{D}^{31}+26\mathcal{D}^{32}+24\mathcal{D}^{33}+22\mathcal{D}^{34}+21\mathcal{D}^{35}\\ +13\mathcal{D}^{36}+14\mathcal{D}^{37}+7\mathcal{D}^{38}+13\mathcal{D}^{39}+3\mathcal{D}^{40}+11\mathcal{D}^{41}+\mathcal{D}^{42}+9\mathcal{D}^{43}+\mathcal{D}^{44}+7\mathcal{D}^{45}+5\mathcal{D}^{47}+2\mathcal{D}^{49}+\mathcal{D}^{59}\end{array}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)^3(1-\mathcal{D}^{10})^2(1-\mathcal{D}^{12})}$
4	$\dfrac{\begin{array}{c}1+3\mathcal{D}^{10}+6\mathcal{D}^{12}+11\mathcal{D}^{14}+17\mathcal{D}^{16}+22\mathcal{D}^{18}+31\mathcal{D}^{20}+36\mathcal{D}^{22}+48\mathcal{D}^{24}+53\mathcal{D}^{26}+58\mathcal{D}^{28}\\ +58\mathcal{D}^{30}+48\mathcal{D}^{32}+38\mathcal{D}^{34}+23\mathcal{D}^{36}+14\mathcal{D}^{38}+6\mathcal{D}^{40}+4\mathcal{D}^{42}+2\mathcal{D}^{44}+\mathcal{D}^{46}\end{array}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)^3(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
3	$\dfrac{\begin{array}{c}1+\mathcal{D}^8+2\mathcal{D}^9+2\mathcal{D}^{10}+2\mathcal{D}^{11}+3\mathcal{D}^{12}+5\mathcal{D}^{13}+4\mathcal{D}^{14}+6\mathcal{D}^{15}+5\mathcal{D}^{16}+6\mathcal{D}^{17}+6\mathcal{D}^{18}+6\mathcal{D}^{19}+5\mathcal{D}^{20}+6\mathcal{D}^{21}+6\mathcal{D}^{22}\\ +5\mathcal{D}^{23}+6\mathcal{D}^{24}+6\mathcal{D}^{25}+6\mathcal{D}^{26}+5\mathcal{D}^{27}+6\mathcal{D}^{28}+4\mathcal{D}^{29}+5\mathcal{D}^{30}+3\mathcal{D}^{31}+2\mathcal{D}^{32}+2\mathcal{D}^{33}+2\mathcal{D}^{34}+\mathcal{D}^{35}+\mathcal{D}^{43}\end{array}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
2	$\dfrac{1+\mathcal{D}^4+\mathcal{D}^6+2\mathcal{D}^8+\mathcal{D}^{10}+3\mathcal{D}^{12}+3\mathcal{D}^{16}+\mathcal{D}^{18}+\mathcal{D}^{22}}{(1-\mathcal{D}^8)(1-\mathcal{D}^{12})^2}$

Table 5.2: The Hilbert series for a singe scalar field, with a fixed field content $\Phi^{\mathbf{6}}$.

		$d_{\mathbf{4}n}$																				
d	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
≥ 4		1	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	2	0	2
3		1	0	0	0	1	0	1	0	1	1	1	0	2	1	1	1	2	1	2	1	2
2		1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1

Table 5.3: Coefficients of the Hilbert series for a singe scalar field, with a fixed field content $\Phi^{\mathbf{4}}$.

		$d_{\mathbf{5}n}$																			
d	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
≥ 5		1	0	0	0	1	0	1	0	2	0	2	0	5	0	4	0	8	0	9	0	13
4		1	0	0	0	1	0	1	0	2	0	3	0	5	0	6	0	10	0	12	0	17
3		1	0	0	0	1	0	1	0	1	1	2	0	4	2	3	3	6	3	7	5	8
2		1	0	0	0	1	0	0	0	1	0	0	0	3	0	0	0	3	0	0	0	3

Table 5.4: Coefficients of the Hilbert series for a singe scalar field, with a fixed field content $\Phi^{\mathbf{5}}$.

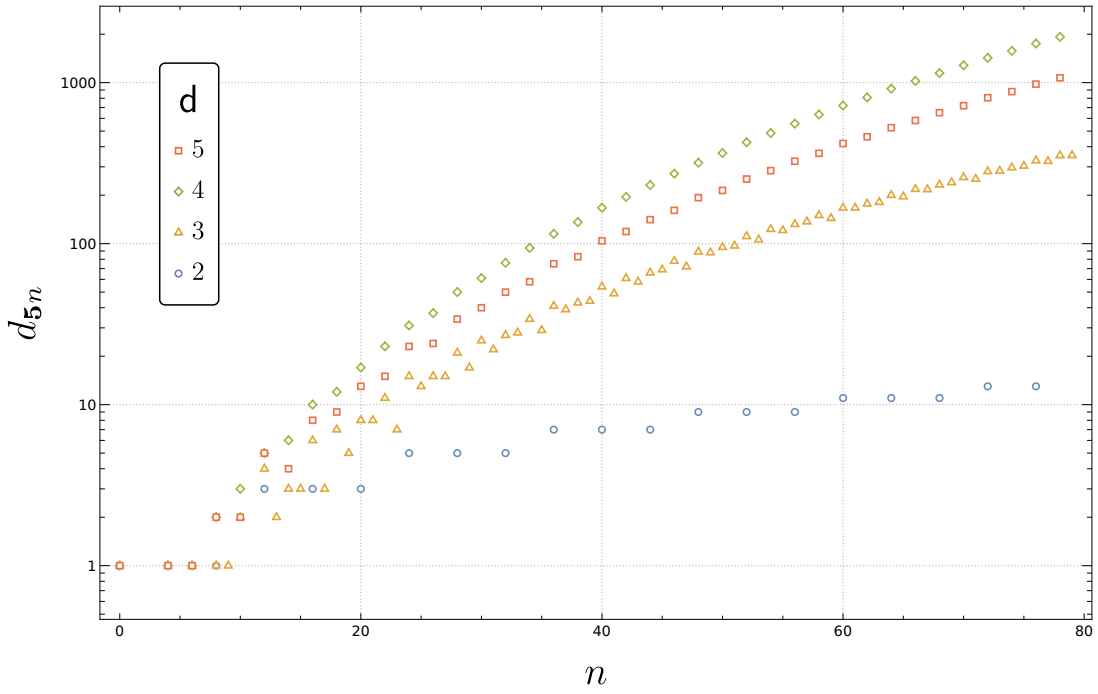


Figure 5.1: Semi-log plot of $d_{\mathbf{5}n}$, that is the number of independent operators of the type $\partial^n\phi^{\mathbf{5}}$. For dimensions $\mathsf{d} = 2, \ldots, 5$.

Remark 5.6 (Analysis of $H_{\mathbf{5}}(\mathcal{D})$)**.** Now turning our attention to $H_{\mathbf{5}}(\mathcal{D})$, we see even higher level of complexity. Not only we have more terms in the denominator, reflecting more *building blocks* of how one can assign and contract derivatives, but also even in $\mathsf{d} \geq 5$ where we can not use $\boldsymbol{\varepsilon}$, we have more *starting blocks*. This simple interpretation of the numerator as extra starting operators is possible in general for a scalar field [1].

However, in a more complicated EFT we can encounter Hilbert series which can not be brought to the canonical form (see Remark 2.12) with only positive terms in the numerator. If this is the case, and we will encounter it in Section 5.3, it is an indication of more complicated relations between operators.

Comparing $H_{\mathbf{5}}(\mathcal{D})$ for $\mathsf{d} = 4$ and $\mathsf{d} = 5$ we can see, that in $\mathsf{d} = 4$ we have several (similarly as in discussion about $H_{\mathbf{4}}(\mathcal{D})$) additional operators for every one in $\mathsf{d} = 5$. For example, the term $\mathcal{D}^{10}$ corresponds to the operator

$$\boldsymbol{\varepsilon}^{abcd}(\partial\overline{\partial\partial\partial_a\Phi)(\partial\partial\partial_b\Phi)}(\partial\partial_c\Phi)(\partial_d\Phi)(\Phi).$$

In $\mathsf{d} = 3$ we have some similar "starting" operators (just check that some terms in the numerator are same for $\mathsf{d} = 3$ and $\mathsf{d} = 4$, or even $\mathsf{d} = 5$), but others dropped one $\mathcal{D}$, which corresponds to $\boldsymbol{\varepsilon}$ with one index less.

		$d_{\mathbf{6}n}$																			
d	n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
≥ 6		1	0	0	0	1	0	2	0	4	0	6	0	13	0	19	0	36	0	58	0	97
5		1	0	0	0	1	0	2	0	4	0	6	0	13	0	19	1	36	1	58	4	97
4		1	0	0	0	1	0	2	0	4	0	6	0	14	0	23	0	45	0	75	0	132
3		1	0	0	0	1	0	2	0	3	2	5	2	10	7	13	12	23	19	35	32	50
2		1	0	0	0	1	0	1	0	3	0	1	0	6	0	1	0	8	0	4	0	10

Table 5.5: Coefficients of the Hilbert series for a singe scalar field, with a fixed field content $\Phi^{\mathbf{6}}$.

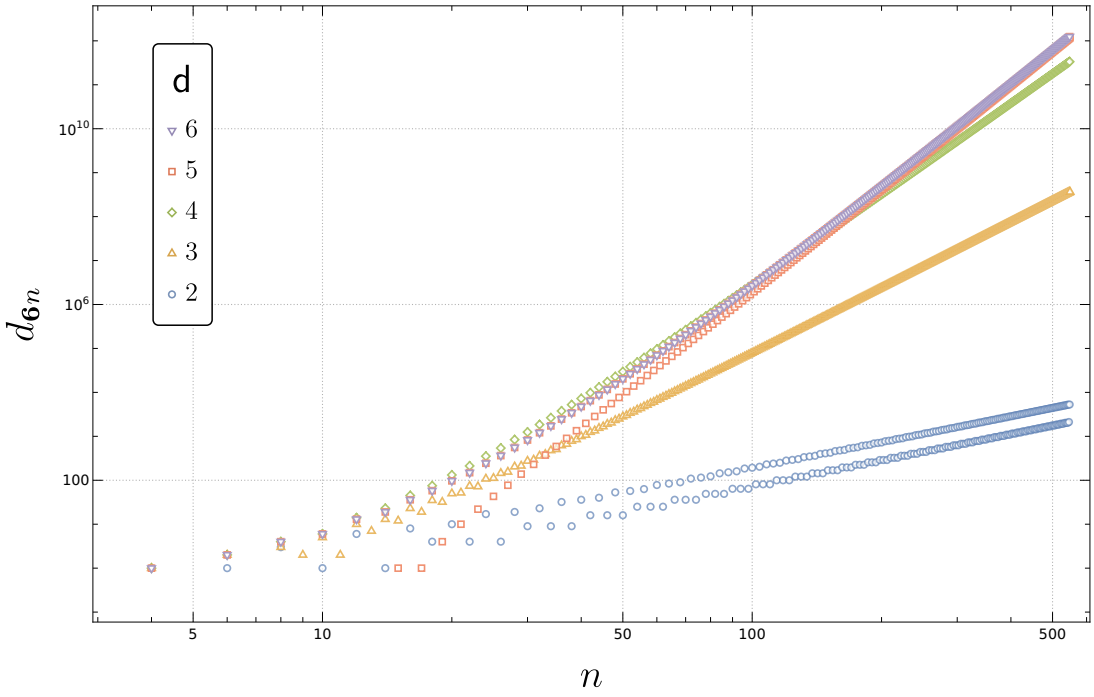


Figure 5.2: Log-log plot of the coefficients of $H_{\mathbf{6}}(\mathcal{D})$ in $\mathsf{d} = 2, \ldots, 6$.

Remark 5.7 (Analysis of $H_{\mathbf{6}}(\mathcal{D})$)**.** Although looking at Table 5.5 it seems there are more operators in $\mathsf{d} = 4$, at around $n \approx 100$ dimensions $\mathsf{d} = 5, 6$ overtake the lead (see Figure 5.2). For even n we have $d_{\mathbf{6}n}^{\mathsf{d}=5} = d_{\mathbf{6}n}^{\mathsf{d}=6}$, but in $\mathsf{d} = 5$ we additionally have odd operators containing $\boldsymbol{\varepsilon}^{\bullet\cdots\bullet}$.

Going down with d, the number of terms in the denominator decreases due to GDC relations, leading to *exponential* suppression of the number of independent operators with given $\mathcal{D}$. Asymptotics of operator Hilbert series were studied by Melia and Pal [15]. Perhaps surprisingly, they seem to approximate pretty well even for small n.

5.2 Sidenote — Gram determinant conditions

As we already mentioned several times, when lowering the dimension d some operators can vanish, as is evident from Tables 5.1 and 5.2. We will give an explicit example of such operator and make some comments about this phenomenon.

The simplest example we already pointed out were operators of the type $\partial^6\Phi^{\mathbf{4}}$, which somehow vanish in $\mathsf{d}=2$. We will now show that EOM conditions are enough to eliminate one of the operators written out in Remark 5.3. Just expanding contractions, using the EOM relation (in $\mathsf{d}=2$) for green terms

$$\overbracket{\partial\partial}\Phi = \partial^a\partial_a\Phi \equiv \partial^1\partial_1\Phi + \partial^2\partial_2\Phi \overset{!}{=} 0 \quad\Longrightarrow\quad \partial^1\partial_1\Phi = -\partial^2\partial_2\Phi,$$

and commuting some of the second derivatives leads to (terms cancel pair by pair)

$$\overbracket{\partial\partial\Phi\partial\partial\Phi\partial\partial}\Phi\Phi = \begin{matrix}+\partial^1\partial_1\Phi\partial^1\partial_1\Phi\partial^1\partial_1\Phi\Phi\\ +\partial^2\partial_1\Phi\partial^1\partial_1\Phi\partial^1\partial_2\Phi\Phi\\ +\partial^1\partial_2\Phi\partial^2\partial_1\Phi\partial^1\partial_1\Phi\Phi\\ +\partial^2\partial_2\Phi\partial^2\partial_1\Phi\partial^1\partial_2\Phi\Phi\\ +\partial^1\partial_1\Phi\partial^1\partial_2\Phi\partial^2\partial_1\Phi\Phi\\ +\partial^2\partial_1\Phi\partial^1\partial_2\Phi\partial^2\partial_2\Phi\Phi\\ +\partial^1\partial_2\Phi\partial^2\partial_2\Phi\partial^2\partial_1\Phi\Phi\\ +\partial^2\partial_2\Phi\partial^2\partial_2\Phi\partial^2\partial_2\Phi\Phi\end{matrix} = \begin{matrix}+\partial^1\partial_1\Phi\partial^1\partial_1\Phi\partial^1\partial_1\Phi\Phi\\ +\partial^2\partial_1\Phi\partial^1\partial_1\Phi\partial^1\partial_2\Phi\Phi\\ +\partial^1\partial_2\Phi\partial^2\partial_1\Phi\partial^1\partial_1\Phi\Phi\\ +\partial^2\partial_2\Phi\partial^2\partial_1\Phi\partial^1\partial_2\Phi\Phi\\ -\partial^2\partial_2\Phi\partial^1\partial_2\Phi\partial^2\partial_1\Phi\Phi\\ -\partial^2\partial_1\Phi\partial^1\partial_2\Phi\partial^1\partial_1\Phi\Phi\\ -\partial^1\partial_2\Phi\partial^1\partial_1\Phi\partial^2\partial_1\Phi\Phi\\ -\partial^1\partial_1\Phi\partial^1\partial_1\Phi\partial^1\partial_1\Phi\Phi\end{matrix} = 0.$$

But where are the celebrated GDC conditions? Attentive readers may recognize, that we encountered very similar operator already in Section 2.1. Just by appending one Φ, by the same manipulations we obtain a Gram determinant condition

$$0 \overset{!}{=} \partial^{[a}\partial_a\Phi\partial^b\partial_b\Phi\partial^{c]}\partial_c\Phi\Phi = \overbracket{\partial\partial}\Phi\overbracket{\partial\partial}\Phi\overbracket{\partial\partial}\Phi\Phi + 2\overbracket{\partial\partial\Phi\partial\partial\Phi\partial\partial}\Phi\Phi - 3\overbracket{\partial\partial\Phi\partial\partial}\Phi\overbracket{\partial\partial}\Phi\Phi.$$

Since the first and the third term on the right side are automatically zero by EOM, the second term must necessarily vanish in $\mathsf{d}=2$ "just" by EOM.

Essentially the same thing happens for $H_{\mathbf{6}}(\mathcal{D})$ (see Table 5.2) when going down from $\mathsf{d}=5$ to $\mathsf{d}=4$, or from $\mathsf{d}=4$ to $\mathsf{d}=3$. It would be interesting to understand what happens from $\mathsf{d}=3$ to $\mathsf{d}=2$, where we lose $(1-\mathcal{D}^4)(1-\mathcal{D}^6)^2(1-\mathcal{D}^{10})$, but instead gain $(1-\mathcal{D}^{12})$ in the denominator, which is rather peculiar. As of now, we do not have any explanation.

On the other hand, it is interesting to study when GDC do not play a role, either when there are none or they are automatically satisfied. In the special case of scalar fields, such conditions can only exist for $\mathsf{d} < \boldsymbol{r}$, otherwise it is not possible to construct a trivial operator that is non-trivial in some higher dimension. This is because we need to antisymmetrize in at least $\mathsf{d}+1$ indices, but since derivatives commute and Φ do not have any indices, the only possibility is to antisymmetrize indices of partial derivatives standing before different fields Φ.

One would then expect some GDC relations manifesting already for $\mathsf{d}+1=\boldsymbol{r}$, but quick examination of Tables 5.1 and 5.2 shows otherwise. This is a consequence of IBP relations, which admit us to move all derivatives in front of one Φ away, thus we are able to find at most d indices in which we can antisymmetrize. Therefore, GDC conditions in the presence of IBP become effective for $\mathsf{d}+2 \leq \boldsymbol{r}$.

Preceding discussion holds in general for scalar fields, but things can get more complicated very quick. For example see the next section.

5.3 Photons, electromagnetic field

After having some experience with the scalar field, we can try to apply our formalism to slightly more complicated case of electromagnetic field. This section is mainly based on the work of Chowdhury *et al.* [16].

It is advantageous to choose the Maxwell tensor $F_{ab} \equiv \mathrm{d}_a A_b = 2\partial_{[a} A_{b]}$ as a building block of the Lagrangian density instead of the vector potential A_b, since it automatically accounts for the gauge invariance. The only essentially new thing will be the construction of the *single particle graded representation* R_F, and calculation of the corresponding graded character χ_{R_F}.

Calculation 5.8 (EOM and other relations)**.** First, we must understand the structure of the building block F_{ab}. The kinetic Lagrangian density has the well-known form

$$\mathcal{L}_{\mathrm{kin}}(A_b, \partial_a A_b) \equiv -\frac{1}{4} F_{ab} F^{ab},$$

to which correspond free equations of motion (the first set of *Maxwell's equations*)

$$\partial^a F_{ab} = 0. \tag{\textbf{EOM}}$$

Another relation F_{ab} automatically satisfies from its definition as exterior derivative of A_b are so-called *Bianchi identities* (the second set of *Maxwell's equations*)

$$3\partial_{[a} F_{bc]} = \mathrm{d}_a F_{bc} = \mathrm{d}_a \mathrm{d}_b A_c = 0. \tag{\textbf{Bianchi}}$$

One relation we will use is already a consequence of previous two, because

$$\partial^a \partial_a F_{bc} = \partial^a \partial_a F_{bc} + \cancel{\partial^a \partial_c F_{ab}}^{\text{by (EOM)}} + \cancel{\partial^a \partial_b F_{ca}}^{\text{by (EOM)}} = 3\partial^a \partial_{[a} F_{bc]} \overset{\text{by (Bianchi)}}{=\!=} 0.$$

Remark 5.9. The Hilbert series for the electromagnetic field will also include the kinetic term (because we work with $F_{\bullet\bullet}$ instead of $A_\bullet$), whereas in the case of a scalar field the kinetic term is always eliminated by IBP and EOM.

Also note that we must make some corrections by hand, namely account for Chern-Simons topological terms of type $A \wedge F \wedge \cdots \wedge F$, or overcounting of $F \wedge F \wedge \ldots \wedge F = \mathrm{d}(A \wedge \mathrm{d}A \wedge \ldots \wedge \mathrm{d}A)$. But more on that later.

Now we would like to build R_F by repeatedly applying derivatives on F_{ab}, but we also need to continuously utilize all possible relations to avoid any redundancies. It is therefore useful to decompose representations we obtain to smaller pieces, some of which will be zero by usage of the relations.

Calculation 5.10 (Direct "manual" decomposition of $\partial_a F_{bc}$ using indices)**.** Using antisymmetry of F_{ab} we can perform manipulations

$$\begin{aligned}
\partial_a F_{bc} + \overbrace{\partial_c F_{ab}}^{0} + \overbrace{\partial_b F_{ca}}^{0} \quad & 3\partial_{[a} F_{bc]} \quad & 3\partial_{[a} F_{bc]} \\
3\partial_a F_{[bc]} = \partial_a F_{bc} \quad + \partial_b F_{ac} = \quad & 2\partial_{(a} F_{b)c} = \quad & 2\partial_{\{a} F_{b\}c} + \tfrac{2}{\mathsf{d}} \boldsymbol{\delta}_{ab} \partial^d F_{dc} \\
\partial_a F_{bc} + \partial_c F_{ba} \quad - & 2\partial_{(a} F_{c)b} \quad & - 2\partial_{\{a} F_{c\}b} - \tfrac{2}{\mathsf{d}} \boldsymbol{\delta}_{ac} \partial^d F_{db},
\end{aligned}$$

where $\{\bullet\bullet\}$ denotes the traceless symmetric part, thus obtaining decomposition

$$\partial_a F_{[bc]} = \partial_{[a} F_{bc]} + \frac{4}{3}\partial_{\{a} F_{[b\}c]} - \frac{4}{3\mathsf{d}}\boldsymbol{\delta}_{a[b}\partial^d F_{c]d},$$

where it is understood that first is performed the antisymmetrization $[bc]$ and only after that the traceless symmetrization $\{\bullet\bullet\}$ in the corresponding indices. The first fully antisymmetric term is zero by (**Bianchi**) and the contraction in the last term is zero by (**EOM**).

Remark 5.11 (Young diagrams and symmetrizers)**.** Recalling Remark 4.22, we can efficiently describe this decomposition by usage of *Young diagrams*. Since the theory is rather complicated, in the following we will work just intuitively. We assign to every index one box, and represent their symmetries by a certain left- and top-justified diagram. For example, to fully symmetric traceless tensor and fully antisymmetric tensor we assign diagrams

$$T_{\{a_1a_2\ldots a_n\}} \longleftrightarrow \boxed{1}\boxed{2}\cdots\boxed{n} \qquad \text{and} \qquad T_{[a_1a_2\ldots a_n]} \longleftrightarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline n\\\hline\end{array}.$$

More complicated *mixed* symmetries (irreducible under $\mathfrak{S}_n$) are represented with diagrams of multiple rows and columns, for example

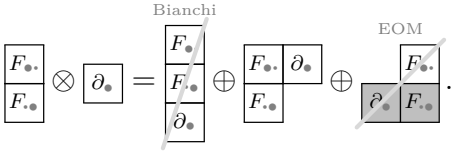
$$\partial_{\{a} F_{[b\}c]} \longleftrightarrow \begin{array}{|c|c|}\hline b & a\\\hline c\\\cline{1-1}\end{array},$$

where the Young diagram λ is always interpreted as follows:

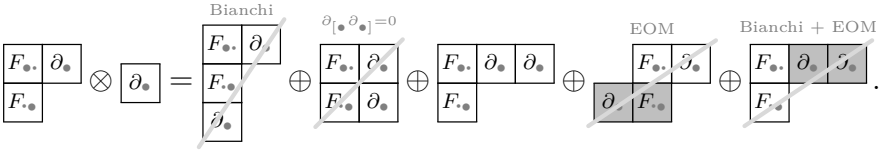
- First perform all permutations preserving each column of λ, and sum over them multiplied with the corresponding sign of permutation.
- Then perform all permutations preserving each row of λ and sum over them.

This is essentially the definition of *Young symmetrizer* corresponding to λ.

Calculation 5.12 (Decomposition using Young diagrams)**.** To decompose tensor of the form $\lambda \otimes \square$, we essentially make all available antisymmetric products and all available symmetric products, which we further decompose to traceless symmetric product and the corresponding traces. Leaving out the traces, this can be represented by attaching the box $\square$ to all possible places of the Young diagram λ. Thus, the decomposition performed earlier directly using the indices can be done diagrammatically as (gray boxes represent contraction)



Following the prescription, we easily find the decomposition of $\partial_a\partial_{\{b}F_{[c\}d]}$ as



Calculation 5.13 (Single Particle Graded Representation R_F for electromagnetic field)**.** Continuing further, same arguments give us that any trace is zero by (**EOM**) and any antisymmetrization containing derivative is zero by (**Bianchi**) or symmetry of derivatives, therefore we obtain R_F of the form

$$R_F = \mathrm{Span}\begin{pmatrix} F_{[ab]} \\ \partial_{\{a_1} F_{[a\}b]} \\ \partial_{\{a_1}\partial_{a_2} F_{[a\}b]} \\ \vdots \\ \partial_{\{a_1}\cdots\partial_{a_n} F_{[a\}b]} \\ \vdots \end{pmatrix} \equiv \mathcal{D}\,\ydiagram{1,1} \oplus \mathcal{D}^2\,\ydiagram{2,1} \oplus \mathcal{D}^2\,\ydiagram{3,1} \oplus \cdots \oplus \mathcal{D}^n\,\overbrace{\ydiagram{2,1}\cdots}^{n} \oplus \cdots$$

with the graded character (because $F_{ab} = \mathrm{d}_a A_b$ already contains one derivative)

$$\chi_{R_F}(\mathcal{D};\boldsymbol{x}) \equiv \mathcal{D}\chi_{\ydiagram{1,1}}(\boldsymbol{x}) + \mathcal{D}^2\chi_{\ydiagram{2,1}}(\boldsymbol{x}) + \mathcal{D}^3\chi_{\ydiagram{3,1}}(\boldsymbol{x}) + \mathcal{D}^3\chi_{\ydiagram{4,1}}(\boldsymbol{x}) + \cdots .$$

Calculation 5.14 (Graded character of R_F)**.** Casting also the SPGR R_Φ for a single scalar field (see Derivation 4.6) in the language of Young diagrams takes the form (we denote the trivial representation by $\bigcirc$)

$$R_\Phi \equiv S^{\{\bullet\}}(\square) = \bigcirc \oplus \mathcal{D}\,\ydiagram{1} \oplus \mathcal{D}^2\,\ydiagram{2} \oplus \cdots \oplus \mathcal{D}^n\,\overbrace{\ydiagram{2}\cdots}^{n} \oplus \cdots$$

with the corresponding character

$$\chi_{R_\Phi}(\mathcal{D};\boldsymbol{x}) \equiv \chi_{S^{\{\bullet\}}(\square)}(\mathcal{D};\boldsymbol{x}) = 1 + \mathcal{D}\chi_{\ydiagram{1}}(\boldsymbol{x}) + \mathcal{D}^2\chi_{\ydiagram{2}}(\boldsymbol{x}) + \mathcal{D}^3\chi_{\ydiagram{3}}(\boldsymbol{x}) + \cdots .$$

Using Proposition 3.49 and some reordering we can calculate

$$\begin{aligned}
\chi_{R_\Phi}(\mathcal{D};\boldsymbol{x})\Big(\mathcal{D}\chi_{\ydiagram{1}}(\boldsymbol{x})\Big) &= \Big(1 + \mathcal{D}\chi_{\ydiagram{1}}(\boldsymbol{x}) + \mathcal{D}^2\chi_{\ydiagram{2}}(\boldsymbol{x}) + \mathcal{D}^3\chi_{\ydiagram{3}}(\boldsymbol{x}) + \cdots\Big)\Big(\mathcal{D}\chi_{\ydiagram{1}}(\boldsymbol{x})\Big) \\
&= \mathcal{D}\chi_{\ydiagram{1}}(\boldsymbol{x}) + \mathcal{D}^2\chi_{\ydiagram{1}\otimes\ydiagram{1}}(\boldsymbol{x}) + \mathcal{D}^3\chi_{\ydiagram{2}\otimes\ydiagram{1}}(\boldsymbol{x}) + \mathcal{D}^4\chi_{\ydiagram{3}\otimes\ydiagram{1}}(\boldsymbol{x}) + \cdots \\
&= \mathcal{D}\chi_{\ydiagram{1}}(\boldsymbol{x}) + \mathcal{D}^2\chi_{\ydiagram{2}}(\boldsymbol{x}) \;+ \mathcal{D}^3\chi_{\ydiagram{3}}(\boldsymbol{x}) \;+ \mathcal{D}^4\chi_{\ydiagram{4}}(\boldsymbol{x}) \;+ \cdots \\
&\qquad + \mathcal{D}^2\chi_{\ydiagram{1,1}}(\boldsymbol{x}) \;+ \mathcal{D}^3\chi_{\ydiagram{2,1}}(\boldsymbol{x}) \;+ \mathcal{D}^4\chi_{\ydiagram{3,1}}(\boldsymbol{x}) \;+ \cdots \\
&\qquad + \mathcal{D}^2 \;+ \mathcal{D}^3\chi_{\ydiagram{1}}(\boldsymbol{x}) \;+ \mathcal{D}^4\chi_{\ydiagram{2}}(\boldsymbol{x}) \;+ \cdots \\
&= \Big(\chi_{R_\Phi}(\mathcal{D};\boldsymbol{x}) - 1\Big) + \mathcal{D}\chi_{R_F}(\mathcal{D};\boldsymbol{x}) + \mathcal{D}^2\chi_{R_\Phi}(\mathcal{D};\boldsymbol{x}),
\end{aligned}$$

where in the third equality the first line is the traceless symmetrized product, the second line is the antisymmetric product, and the third line is the trace. We can thus express the graded character of R_F as

$$\begin{aligned}
\chi_{R_F}(\mathcal{D},\boldsymbol{x}) &= \frac{\Big(\mathcal{D}\chi_{\ydiagram{1}}(\boldsymbol{x}) - (1+\mathcal{D}^2)\Big)\chi_{R_\Phi}(\mathcal{D};\boldsymbol{x}) + 1}{\mathcal{D}} \\
&= \frac{\Big((\mathcal{D}-\mathcal{D}^3)\chi_{\ydiagram{1}}(\boldsymbol{x}) - (1-\mathcal{D}^4)\Big)P(\mathcal{D};\boldsymbol{x}) + 1}{\mathcal{D}}
\end{aligned}$$

where the character $\chi_{\ydiagram{1}}$ is given in Example 3.92.

To calculate the Hilbert series $H_{F^r}(\mathcal{D})$ with a fixed field content $F^{\boldsymbol{r}}$, we proceed exactly as in Section 5.1 just with the substitution $\chi_{R_\Phi}(\mathcal{D};\boldsymbol{x}) \mapsto \chi_{R_F}(\mathcal{D};\boldsymbol{x})$. We list some of the results in Table 5.6, partly reproducing and partly extending calculations of Henning, Lu, Melia, and Murayama [1] and Chowdhury *et al.* [16].

d	$\frac{1}{\mathcal{D}^4}H_{F^4}(\mathcal{D})$	miscount	$\frac{1}{\mathcal{D}^5}H_{0,F^5}(\mathcal{D})$
≥ 10	$\dfrac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\cdots$
9	$\dfrac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$-\mathcal{D}$	$\cdots$
8	$\dfrac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$+1$	$\cdots$
7	$\dfrac{\frac{1}{\mathcal{D}}+2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$	$-\frac{1}{\mathcal{D}}$	$\dfrac{\mathcal{D}^3+4\mathcal{D}^4+4\mathcal{D}^5+16\mathcal{D}^6+10\mathcal{D}^7+39\mathcal{D}^8+17\mathcal{D}^9+69\mathcal{D}^{10}+28\mathcal{D}^{11}+99\mathcal{D}^{12}+36\mathcal{D}^{13}+125\mathcal{D}^{14}+41\mathcal{D}^{15}+135\mathcal{D}^{16}+43\mathcal{D}^{17}+126\mathcal{D}^{18}+38\mathcal{D}^{19}+105\mathcal{D}^{20}+28\mathcal{D}^{21}+73\mathcal{D}^{22}+19\mathcal{D}^{23}+41\mathcal{D}^{24}+10\mathcal{D}^{25}+19\mathcal{D}^{26}+2\mathcal{D}^{27}+5\mathcal{D}^{28}-2\mathcal{D}^{30}-\mathcal{D}^{31}-2\mathcal{D}^{32}-\mathcal{D}^{33}+\mathcal{D}^{37}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
6	$\dfrac{2+3\mathcal{D}^2+2\mathcal{D}^4+\mathcal{D}^8}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\dfrac{1+4\mathcal{D}^2+13\mathcal{D}^4+34\mathcal{D}^6+73\mathcal{D}^8+121\mathcal{D}^{10}+168\mathcal{D}^{12}+210\mathcal{D}^{14}+226\mathcal{D}^{16}+213\mathcal{D}^{18}+182\mathcal{D}^{20}+131\mathcal{D}^{22}+79\mathcal{D}^{24}+42\mathcal{D}^{26}+16\mathcal{D}^{28}+\mathcal{D}^{30}-\mathcal{D}^{32}-\mathcal{D}^{36}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
5	$\dfrac{2+3\mathcal{D}^2+2\mathcal{D}^4}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\dfrac{\mathcal{D}+5\mathcal{D}^3+4\mathcal{D}^4+16\mathcal{D}^5+16\mathcal{D}^6+30\mathcal{D}^7+36\mathcal{D}^8+51\mathcal{D}^9+63\mathcal{D}^{10}+73\mathcal{D}^{11}+89\mathcal{D}^{12}+92\mathcal{D}^{13}+110\mathcal{D}^{14}+103\mathcal{D}^{15}+117\mathcal{D}^{16}+103\mathcal{D}^{17}+108\mathcal{D}^{18}+91\mathcal{D}^{19}+88\mathcal{D}^{20}+71\mathcal{D}^{21}+59\mathcal{D}^{22}+49\mathcal{D}^{23}+32\mathcal{D}^{24}+27\mathcal{D}^{25}+13\mathcal{D}^{26}+12\mathcal{D}^{27}+2\mathcal{D}^{28}+3\mathcal{D}^{29}-3\mathcal{D}^{30}-2\mathcal{D}^{32}-\mathcal{D}^{33}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
4	$\dfrac{3+5\mathcal{D}^2+\mathcal{D}^4-2\mathcal{D}^6}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\dfrac{2\left(2\mathcal{D}^4+7\mathcal{D}^6+17\mathcal{D}^8+28\mathcal{D}^{10}+35\mathcal{D}^{12}+42\mathcal{D}^{14}+39\mathcal{D}^{16}+28\mathcal{D}^{18}+18\mathcal{D}^{20}+4\mathcal{D}^{22}-7\mathcal{D}^{24}-8\mathcal{D}^{26}-7\mathcal{D}^{28}-7\mathcal{D}^{30}-\mathcal{D}^{32}+2\mathcal{D}^{34}\right)}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^8)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$
3	$\dfrac{1+\mathcal{D}^2+\mathcal{D}^5-\mathcal{D}^6}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)}$		$\dfrac{\mathcal{D}^5+2\mathcal{D}^7+\mathcal{D}^8+\mathcal{D}^9+2\mathcal{D}^{10}+\mathcal{D}^{11}+\mathcal{D}^{12}+\mathcal{D}^{13}+\mathcal{D}^{14}+2\mathcal{D}^{16}-\mathcal{D}^{17}+\mathcal{D}^{25}-\mathcal{D}^{26}}{(1-\mathcal{D}^4)(1-\mathcal{D}^6)(1-\mathcal{D}^{10})(1-\mathcal{D}^{12})}$

Table 5.6: The Hilbert series for an electromagnetic field, with a fixed field content $F^{\mathbf{4}}$ and $F^{\mathbf{5}}$.

Remark 5.15 (Analysis of the highlighted terms in H_{F^4})**.** In the listing of the Hilbert series, we divided by $\mathcal{D}^4$ and $\mathcal{D}^5$, because such number of derivatives is carried automatically by $F^{\mathbf{4}}$ and $F^{\mathbf{5}}$, respectively. It is peculiar that for H_{F^4} in $\mathsf{d}=7$ it seems there is a "starting operator" with only 3 derivatives instead of 4. This is exactly the already mentioned topological Chern-Simons term $A\wedge F\wedge F\wedge F$, which is a top-form that can be integrated over the space-time. The gauge transformation $A\mapsto A+\mathrm{d}u$ only contributes a total derivative term, because

$$\mathrm{d}u\wedge F\wedge F\wedge F=\mathrm{d}u\wedge\mathrm{d}A\wedge\mathrm{d}A\wedge\mathrm{d}A=\mathrm{d}(u\wedge\mathrm{d}A\wedge\mathrm{d}A\wedge\mathrm{d}A).$$

When calculating H_{0,F^4} in $\mathsf{d}=7$ we also obtain additional term of $-\mathcal{D}^3$, which is outside of the fraction. It reflects the fact that the Hilbert series approach does not see $A\wedge F\wedge F\wedge F$ itself, it was able to find operators of the type $\partial^n A\wedge F\wedge F\wedge F$, that is with $n>1$ derivatives somehow distributed between A and Fs, because one derivative can always be borrowed to form $\partial^{n-1}F_{\bullet[\bullet}F_{\bullet\bullet}F_{\bullet\bullet}F_{\bullet\bullet]}$ up to total derivatives. For an example of this procedure see Subsection 5.4 of [16].

Similarly, the miscount $+\mathcal{D}^4$ in $\mathsf{d}=8$ is caused by directly working with F instead of A. The Hilbert series does not realize that the operator $F\wedge F\wedge F\wedge F$ is actually a total derivative, simply $F\wedge F\wedge F\wedge F=\mathrm{d}(A\wedge\mathrm{d}A\wedge\mathrm{d}A\wedge\mathrm{d}A)$.

It is immediately visible that the term $-\mathcal{D}^5$ in $\mathsf{d} = 9$ is a miscount, since it would imply -1 independent operators of the type $\partial F \wedge F \wedge F \wedge F$. It has to do with $*(F \wedge F \wedge F \wedge F)$ being a co-closed one-form, because of Bianchi identities. It is also co-exact, since analogously as before we have

$$*(F \wedge F \wedge F \wedge F) = *\mathrm{d}(A \wedge F \wedge F \wedge F) = \overbrace{*\mathrm{d}*^{-1}}^{\partial\cdot} *(A \wedge F \wedge F \wedge F),$$

but our approach does not "know" about the presence of A, so it miscalculates.

After correcting these miscalculations, it seems we obtain the full Hilbert series H_{F^4}. For construction of the operator basis in $\mathsf{d} = 4$ see [16].

Remark 5.16. The Hilbert series H_{F^4} in $\mathsf{d} = 3, 4$ can not be brought to the canonical form with only positive terms in the numerator (see Remark 5.6). For further discussion on the general structure of the Hilbert series see [1].

Remark 5.17. To our knowledge, results of H_{0,F^5} in $\mathsf{d} = 3, 6, 7$ are *novel*.

5.4 What's next?

We did not even come close to exhausting all interesting EFTs, to which our formalism could be applied. We will just list some of the possibilities:

- We only worked with one particle field at a time, but it is straightforward to consider *multiple particle fields* at once (see Derivation 4.23).
- EFTs of *gravity* are pretty similar to the case of the electromagnetic field, see for example [17] or [16].
- One of the first applications of the Hilbert series approach was done for the *Standard Model EFT* (SMEFT) specially in $\mathsf{d} = 4$ [13]. Gauge invariance was accounted for simply by integration over the gauge group (see Derivation 4.24). Since SMEFT also contains fermions, integration over the covering group $\mathsf{Spin}(4)$ of $\mathsf{SO}(4)$ was necessary, but also easily realized by isomorphism of groups $\mathsf{SU}(2) \times \mathsf{SU}(2) \cong \mathsf{Spin}(4)$.
- From the beginning we restricted ourselves to the case of *linearly realized* symmetries, but for example *Chiral Perturbation Theory* (χPT) does not fall to this category. For a pretty detailed treatment see [1] and [18].

Conclusion

In this thesis we studied operators in the context of Effective Field Theories. Our goal was to compute the Hilbert series encoding information about the number of independent operators of a given type.

Recognizing possible relations between operators — Equations Of Motion (EOM), Integration By Parts (IBP), and Gram Determinant Conditions (GDC) — led us to the study of the representation theory of compact Lie groups.

Along the way we learned handful of things, which can be interesting purely from the mathematical point of view, but they also equipped us with the tools necessary to derive the Master Formula for the Hilbert series. More precisely, we obtained splitting $H = H_0 + \Delta H$, where we found an explicit formula only for the main term H_0. Thankfully, the correction term ΔH typically contains only couple of contributions, which can be calculated by hand.

Finally, we illustrated usage of the formalism in the case of a single scalar and electromagnetic field. With the help of `Mathematica` we were able to compute Hilbert series for various fixed number of particle fields in several dimensions [14]. To our knowledge, some of our results are novel, as we pointed out in the text.

It would be interesting to optimize the code in `Mathematica` as the computational complexity is rather high, since going to higher dimensions or higher number of particle fields results in enormous computation time.

The developed formalism is applicable in a quite general setting. However, there are some aspects we did not have the space to cover adequately, mainly the treatment of fermions (spinors) and inclusion of parity and charge conjugation. The former is just briefly commented on in [1] (but it certainly deserves a deeper discussion), and the latter was thoroughly analyzed in [18].

Unsurprisingly, a lot of questions are waiting to be answered: How to proceed generally if particle fields transform non-linearly under the action of the group? Can we obtain the Hilbert series in a fully closed form? Can we automatize calculation of the correction term ΔH? It just seems, that fun has no limits ...

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